Semidefinite Relaxation of Quadratic Optimization Problems and Applications

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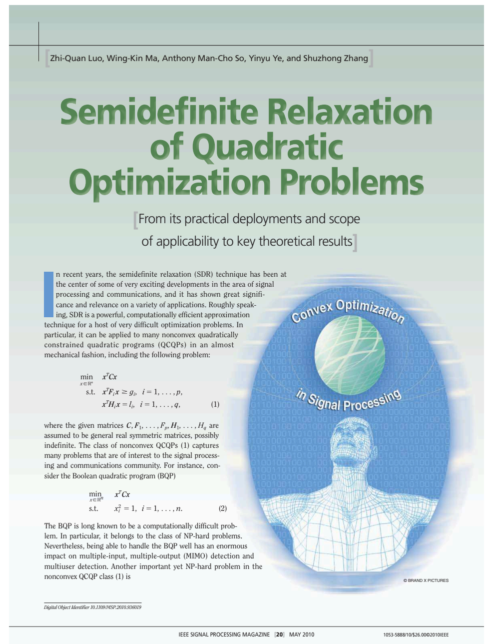
Lesson 9, ELEG5481

- Reference:

  Z.-Q. Luo, W.-K. Ma, Anthony M.-C. So, Y. Ye, & S. Zhang, "Semidefinite relaxation of quadratic optimization problems," in *IEEE SP Magazine*, Special Issue on *Convex Optimization for Signal Process.*, May 2010.



[Zhi-Quan Luo, Wing-Kin Ma, Anthony Man-Cho So, Yinyu Ye, and Shuzhong Zhang]

Semidefinite Relaxation of Quadratic Optimization Problems

[From its practical deployments and scope of applicability to key theoretical results]

In recent years, the semidefinite relaxation (SDR) technique has been at the center of some of very exciting developments in the area of signal processing and communications, and it has shown great significance and relevance on a variety of applications. Roughly speaking, SDR is a powerful, computationally efficient approximation technique for a host of very difficult optimization problems. In particular, it can be applied to many nonconvex quadratically constrained quadratic programs (QCQPs) in an almost mechanical fashion, including the following problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & x^T C x \\ \text{s.t.} \quad & x^T F_i x \geq g_i, \quad i = 1, \ldots, p, \\ & x^T H_i x = l_i, \quad i = 1, \ldots, q, \end{aligned} \qquad (1)$$

where the given matrices $C, F_1, \ldots, F_p, H_1, \ldots, H_q$ are assumed to be general real symmetric matrices, possibly indefinite. The class of nonconvex QCQPs (1) captures many problems that are of interest to the signal processing and communications community. For instance, consider the Boolean quadratic program (BQP)

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & x^T C x \\ \text{s.t.} \quad & x_i^2 = 1, \quad i = 1, \ldots, n. \end{aligned} \qquad (2)$$

The BQP is long known to be a computationally difficult problem. In particular, it belongs to the class of NP-hard problems. Nevertheless, being able to handle the BQP well has an enormous impact on multiple-input, multiple-output (MIMO) detection and multiuser detection. Another important yet NP-hard problem in the nonconvex QCQP class (1) is

Digital Object Identifier 10.1109/MSP.2010.936019

IEEE SIGNAL PROCESSING MAGAZINE [20] MAY 2010 1053-5888/10/$26.00©2010IEEE

- Acknowledgment: Anthony So, Tom Luo, Yinyu Ye, & Shuzhong Zhang.

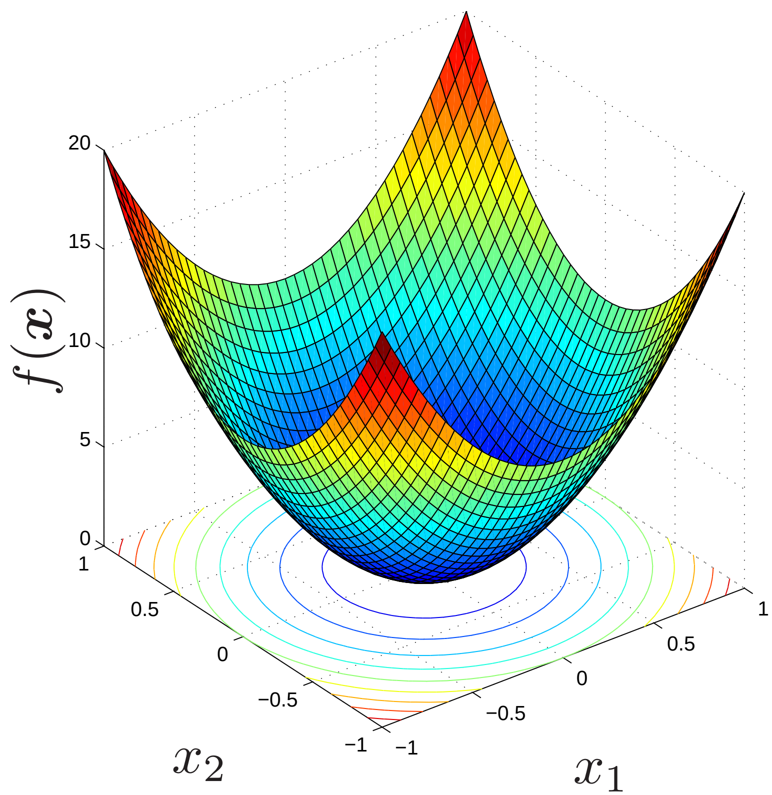
Outline

- Part I: Basic concepts and overview of semidefinite relaxation (SDR)
- Part II: Theory, and implications in practice
- Part III: Frontier Developments
  - Outage-based Transmit Beamforming Optimization

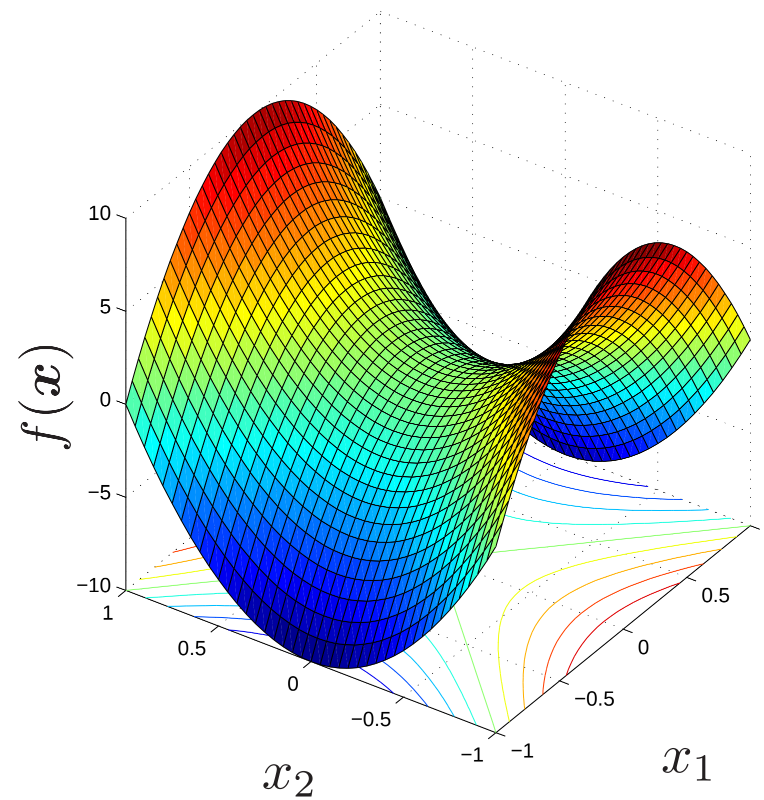
Part I: Basic Concepts and Overview

A quick reminder of what convex quadratic functions & constraints are:

- A function $f(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} = \sum_{i=1}^n \sum_{j=1}^n x_i x_j C_{ij}$ is convex if and only if $\boldsymbol{C} \succeq 0$ ($\boldsymbol{C} \succeq 0$ means that $\boldsymbol{C}$ is positive semidefinite (PSD)).



(a) $\boldsymbol{C} \succeq \boldsymbol{0}$.



(b) $\boldsymbol{C} \not\succeq \boldsymbol{0}$.

Quadratically Constrained Quadratic Program

Consider the class of real-valued **quadratically constrained quadratic programs (QCQPs)**:

$$
\begin{aligned}
\min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{x}^T \boldsymbol{F}_i \boldsymbol{x} \geq g_i, \quad i = 1, \ldots, p, \\
& \boldsymbol{x}^T \boldsymbol{H}_i \boldsymbol{x} = l_i, \quad i = 1, \ldots, q,
\end{aligned}
$$

where $\boldsymbol{C}, \boldsymbol{F}_1, \ldots, \boldsymbol{F}_p, \boldsymbol{H}_1, \ldots, \boldsymbol{H}_q \in \mathbb{S}^n$; $\mathbb{S}^n$ is the set of all $n \times n$ real symmetric matrices.

- We do not consider convex cases, and $\boldsymbol{C}, \boldsymbol{F}_i, \boldsymbol{H}_i$ may be arbitrary.

- Nonconvex QCQP is a very difficult problem in general.

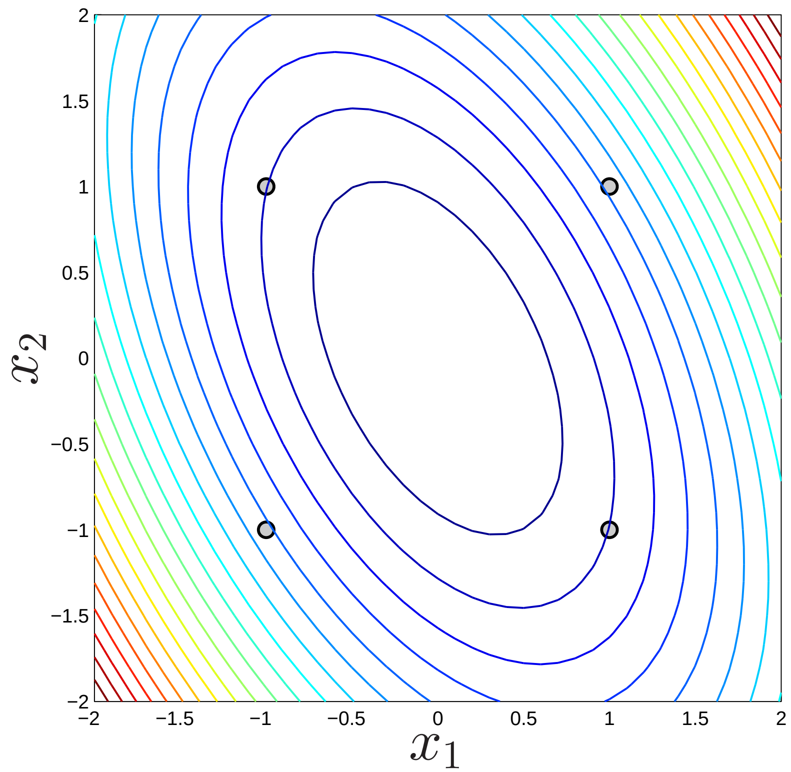
Nonconvex QCQP: How Hard Could it Be?

Consider the **Boolean quadratic program (BQP)**

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x} \\ \text{s.t.} \quad & x_i^2 = 1, \quad i = 1,\ldots,n, \end{aligned}$$

a long-known difficult problem falling in the nonconvex QCQP class.

- You could solve it by evaluating all possible combinations; i.e., brute-force search.
- The complexity of a brute-force search is $\mathcal{O}(2^n)$, not okay at all for large n!
- The BQP is **NP-hard** in general— we still can't find an algorithm that can solve a general BQP in $\mathcal{O}(n^p)$ for any $p > 0$.



Nonconvex QCQP: How Hard Could it Be?

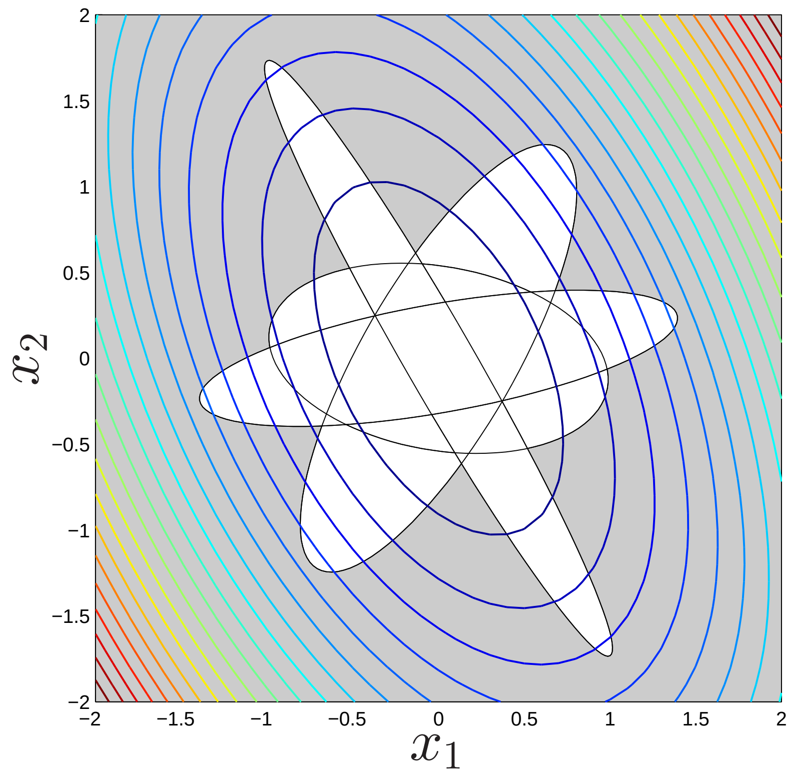
Consider the following problem

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^T \boldsymbol{F}_i \boldsymbol{x} \geq 1, \quad i = 1, \ldots, m, \end{aligned}$$

where $\boldsymbol{C}, \boldsymbol{F}_1, \ldots, \boldsymbol{F}_m$ are all positive semidefinite, or $\boldsymbol{C}, \boldsymbol{F}_1, \ldots, \boldsymbol{F}_m \succeq \mathbf{0}$.

- Difficulty: feasible set is the intersection of the *exteriors* of ellipsoids.

- This problem is also NP-hard.



Semidefinite Relaxation for QCQP

Semidefinite relaxation (SDR) is a computationally efficient approximation approach to QCQP.

- Approximate QCQPs by a **semidefinite program (SDP)**, a class of convex optimization problems where reliable, efficient algorithms are readily available.
- The idea can be found in an early paper of Lovász in 1979 **[Lovász'79]**.
- It is arguably the work by Goemans & Williamson **[Goemans-Williamson'95]** that sparked the significant interest in SDR.
- A key notion introduced by Goemans & Williamson is *randomization*; we will go through that.
- SDR has received much interest in the optimization field; now we have seen a number of theoretically elegant analysis results.
- (This may concern us more) In many applications, SDR works well empirically.

Impacts of SDR in SP and Commun.

- The introduction of SDR in SP and commun. since the early 2000's has reshaped the way we see many topics today.

- Applications identified include
  - multiuser/MIMO detection **[Tan-Rasmussen'01]**, **[Ma-Davidson-Wong-Luo-Ching'02]**
  - multiuser downlink tx beamforming: unicast **[Bengtsson-Ottersten'01]**, multicast **[Sidiropoulos-Davidson-Luo'06]**, &, more recently, multicell downlinks, relaying (incl. analog network coding), cognitive radio, secrecy...
  - sensor network localization **[Biswas-Liang-Wang-Ye'06]**
  - robust blind receive beamforming **[Ma-Ching-Vo'04]**
  - code waveform design in radar **[De Maio *et al.*'08]**
  - transmit B_1 shim in MRI **[Chang-Luo-Wu *et al.*'08]**
  - fusion for distributed detection **[Quan-Ma-Cui-Sayed'10]**
  - binary image restoration, phase unwrapping
  - large-margin parameter estimation in speech recognition **[Li-Jiang'07]**
  - ...

  and the scope of applications is still expanding.

The Concept of SDR

- For notational conciseness, we write the QCQP as

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{x} \trianglerighteq_i b_i, \quad i=1,\ldots,m. \end{aligned} \qquad \text{(QCQP)}$$

  Here, '$\trianglerighteq_i$' can represent either '$\geq$', '$=$', or '$\leq$' for each i; $\boldsymbol{C}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathbb{S}^n$; and $b_1, \ldots, b_m \in \mathbb{R}$.

- A crucial first step of understanding SDR is to see that

$$\boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x} = \mathrm{Tr}(\boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x}) = \mathrm{Tr}(\boldsymbol{C}\boldsymbol{x}\boldsymbol{x}^T), \quad \boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{x} = \mathrm{Tr}(\boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{x}) = \mathrm{Tr}(\boldsymbol{A}_i\boldsymbol{x}\boldsymbol{x}^T),$$

  or, if we let $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T$,

$$\boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x} = \mathrm{Tr}(\boldsymbol{C}\boldsymbol{X}), \quad \boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{x} = \mathrm{Tr}(\boldsymbol{A}_i\boldsymbol{X})$$

- The objective and constraint functions are **linear** in $\boldsymbol{X}$.

The Concept of SDR

- The condition $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T$ is equivalent to $\boldsymbol{X} \succeq \boldsymbol{0}$, $\mathrm{rank}(\boldsymbol{X}) = 1$, thus (QCQP) is the same as

$$\begin{aligned} \min_{\boldsymbol{X} \in \mathbb{S}^n} \quad & \mathrm{Tr}(\boldsymbol{C}\boldsymbol{X}) \\ \text{s.t.} \quad & \mathrm{Tr}(\boldsymbol{A}_i \boldsymbol{X}) \trianglerighteq_i b_i, \quad i = 1, \ldots, m \\ & \boldsymbol{X} \succeq \boldsymbol{0}, \quad \mathrm{rank}(\boldsymbol{X}) = 1. \end{aligned} \tag{QCQP}$$

- The constraints $\mathrm{Tr}(\boldsymbol{A}_i \boldsymbol{X}) \trianglerighteq_i b_i$ are easy, but $\mathrm{rank}(\boldsymbol{X}) = 1$ is hard.

- **Key Insight:** Drop the rank-one constraint to obtain a relaxed QCQP

$$\begin{aligned} \min_{\boldsymbol{X} \in \mathbb{S}^n} \quad & \mathrm{Tr}(\boldsymbol{C}\boldsymbol{X}) \\ \text{s.t.} \quad & \mathrm{Tr}(\boldsymbol{A}_i \boldsymbol{X}) \trianglerighteq_i b_i, \quad i = 1, \ldots, m, \\ & \boldsymbol{X} \succeq \boldsymbol{0}. \end{aligned} \tag{SDR}$$

(SDR) is a convex problem.

Some Merits We Can Immediately Say

- The SDR

$$\begin{aligned} \min_{\boldsymbol{X} \in \mathbb{S}^n} \quad & \mathrm{Tr}(\boldsymbol{C}\boldsymbol{X}) \\ \text{s.t.} \quad & \boldsymbol{X} \succeq \boldsymbol{0}, \quad \mathrm{Tr}(\boldsymbol{A}_i \boldsymbol{X}) \trianglerighteq_i b_i, \quad i = 1, \ldots, m \end{aligned} \tag{SDR}$$

  is a semidefinite program (SDP), whose globally optimal solution may be found by available numerical algorithms in polynomial time (often by interior-point methods, in $\mathcal{O}(\max\{m,n\}^4 n^{1/2} \log(1/\epsilon))$, ϵ being soln. accuracy).

- For instance, using the software toolbox CVX, we can solve (SDR) in MATLAB with the following lines: (for simplicity we assume '$\trianglerighteq_i$' = '$\geq$' for all i here)

```
cvx_begin
    variable X(n,n) symmetric
    minimize(trace(C*X));
    subject to
       for i=1:m
          trace(A(:,:,i)*X) >= b(i);
       end
       X == semidefinite(n)
cvx_end
```

Issues with the Use of SDR

- There is no free lunch in turning the NP-hard (QCQP) to the convex, polynomial-time solvable (SDR).

- The issue is how to convert an SDR solution to an approximate QCQP solution.

- If an SDR solution, say, denoted by $\boldsymbol{X}^\star$, is of rank one; or, equivalently,

$$\boldsymbol{X}^\star = \boldsymbol{x}^\star \boldsymbol{x}^{\star T},$$

  then $\boldsymbol{x}^\star$ is feasible— and in fact optimal— to (QCQP).

- But the case of rank-one SDR solutions does not always hold (otherwise we would have solved an NP-hard problem in polynomial time!)

- There are many ways to produce an approximate QCQP solution from $\boldsymbol{X}^\star$, for instances where $\text{rank}(\boldsymbol{X}^\star) > 1$.

QCQP Solution Approximation in SDR: An Example

- Consider again the BQP

$$\begin{array}{ll} \min & \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} \\ \text{s.t.} & x_i^2 = 1, \quad i = 1, \dots, n. \end{array} \tag{BQP}$$

The SDR of (BQP) is

$$\begin{array}{ll} \min & \text{Tr}(\boldsymbol{C}\boldsymbol{X}) \\ \text{s.t.} & \boldsymbol{X} \succeq \boldsymbol{0}, \quad X_{ii} = 1, \ i = 1, \dots, n. \end{array} \tag{SDR}$$

- An intuitively reasonable idea (true even for engineers) is to apply a rank-1 approximation to the SDR solution $\boldsymbol{X}^\star$:

1) Carry out the eigen-decomposition

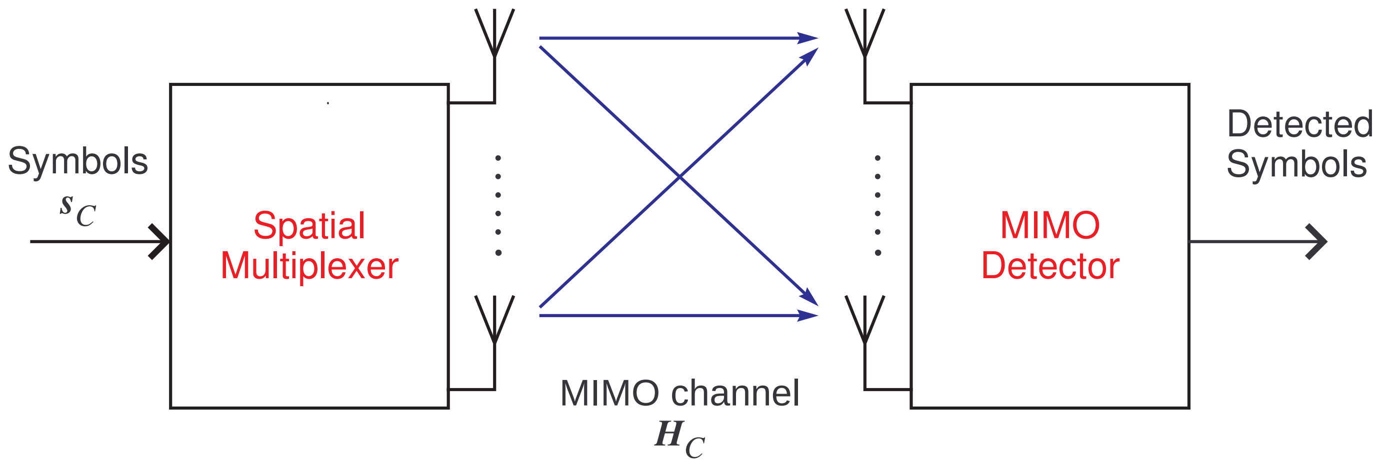
$$\boldsymbol{X}^\star = \textstyle\sum_{i=1}^r \lambda_i \boldsymbol{q}_i \boldsymbol{q}_i^T,$$

where $r = \text{rank}(\boldsymbol{X}^\star)$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$ are the eigenvalues and $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_r \in \mathbb{R}^n$ the respective eigenvectors.

2) Approximate the BQP by $\hat{\boldsymbol{x}} = \text{sgn}(\sqrt{\lambda_1} \boldsymbol{q}_1)$.

Application: MIMO Detection

Scenario: A spatial multiplexing system with M_t transmit & M_r receive antennae.



Objective: detect symbols from the received signals, given channel information.

- Received signal model:

$$\boldsymbol{y}_C = \boldsymbol{H}_C \boldsymbol{s}_C + \boldsymbol{v}_C$$

  where $\boldsymbol{H}_C \in \mathbb{C}^{M_r \times M_t}$ is the MIMO channel, $\boldsymbol{s}_C \in \mathbb{C}^{M_t}$ is the transmitted symbol vector, & $\boldsymbol{v}_C \in \mathbb{C}^{M_r}$ is complex circular Gaussian noise.

- Assume QPSK constellations, $\boldsymbol{s}_C \in \{\pm 1 \pm j\}^{M_t}$.

- Problem: maximum-likelihood (ML) detection (NP-hard)

$$\hat{\boldsymbol{s}}_{C,ML} = \arg \min_{\boldsymbol{s}_C \in \{\pm 1 \pm j\}^{M_t}} \|\boldsymbol{y}_C - \boldsymbol{H}_C \boldsymbol{s}_C\|^2$$

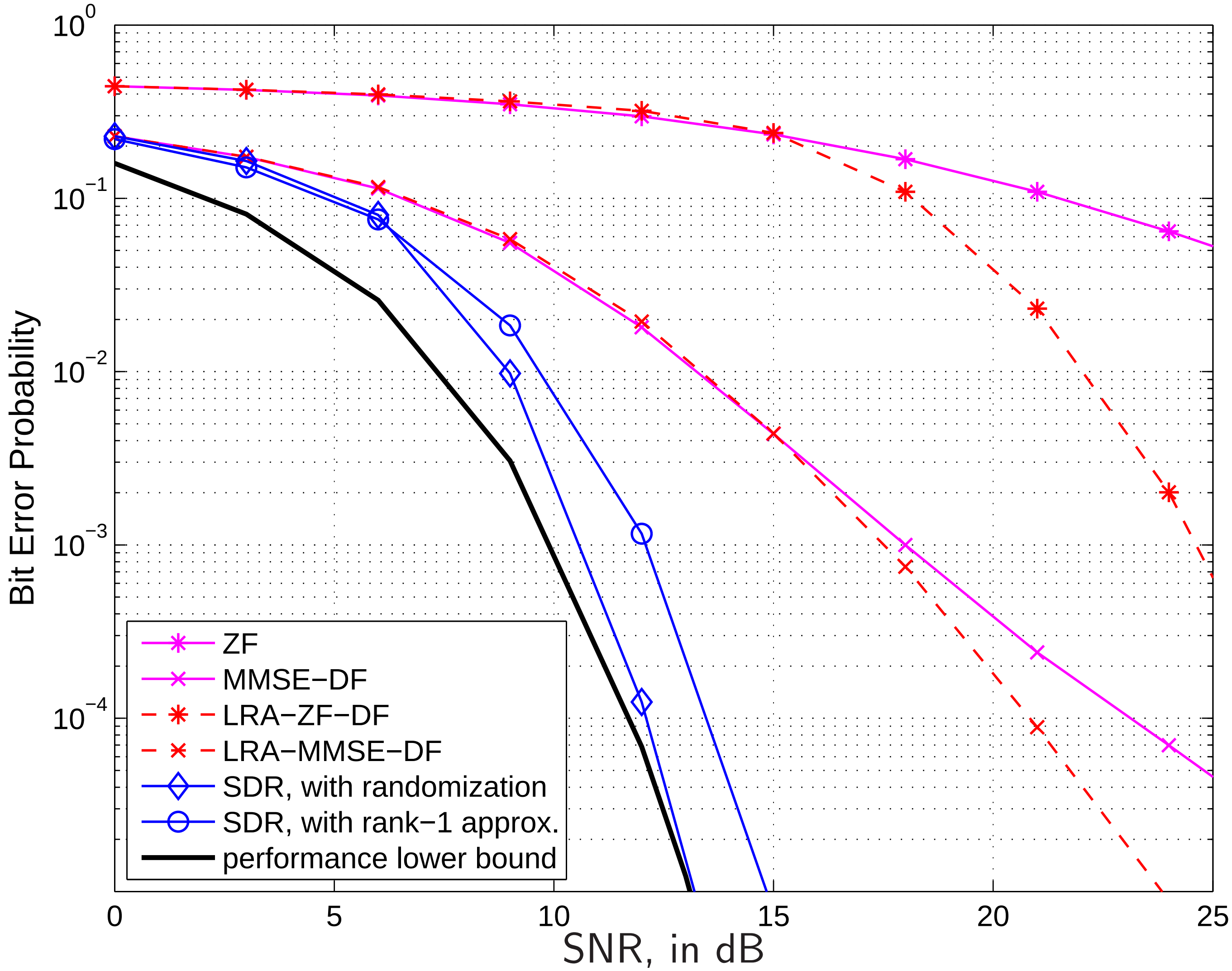
- The received signal model can be converted to a real form

$$\underbrace{\begin{bmatrix} \mathrm{Re}\{\boldsymbol{y}_C\} \\ \mathrm{Im}\{\boldsymbol{y}_C\} \end{bmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{bmatrix} \mathrm{Re}\{\boldsymbol{H}_C\} & -\mathrm{Im}\{\boldsymbol{H}_C\} \\ \mathrm{Im}\{\boldsymbol{H}_C\} & \mathrm{Re}\{\boldsymbol{H}_C\} \end{bmatrix}}_{\boldsymbol{H}} \underbrace{\begin{bmatrix} \mathrm{Re}\{\boldsymbol{s}_C\} \\ \mathrm{Im}\{\boldsymbol{s}_C\} \end{bmatrix}}_{\boldsymbol{s} \in \{\pm 1\}^{2M_t}} + \underbrace{\begin{bmatrix} \mathrm{Re}\{\boldsymbol{v}_C\} \\ \mathrm{Im}\{\boldsymbol{v}_C\} \end{bmatrix}}_{\boldsymbol{v}},$$

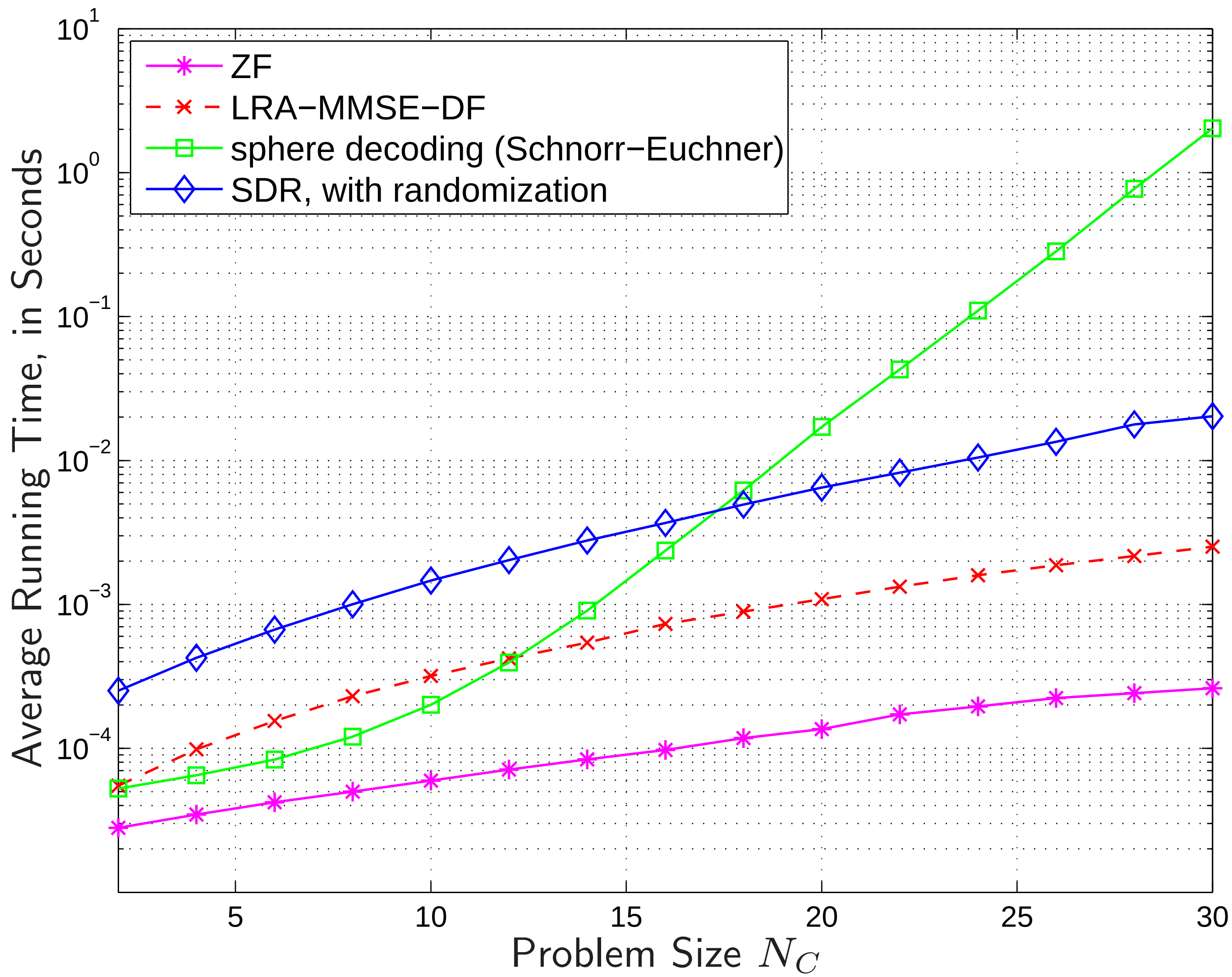
and hence the ML problem can be rewritten (homogenized) as

$$\begin{aligned} \min_{\boldsymbol{s} \in \{\pm 1\}^{2M_t}} \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{s}\|^2 &= \min_{\boldsymbol{s} \in \{\pm 1\}^{2M_t}, t \in \{\pm 1\}} \|t\boldsymbol{y} - \boldsymbol{H}\boldsymbol{s}\|^2 \\ &= \min_{\boldsymbol{s} \in \{\pm 1\}^{2M_t}, t \in \{\pm 1\}} \begin{bmatrix} \boldsymbol{s}^T & t \end{bmatrix} \begin{bmatrix} \boldsymbol{H}^T\boldsymbol{H} & -\boldsymbol{H}^T\boldsymbol{y} \\ -\boldsymbol{y}^T\boldsymbol{H} & \|\boldsymbol{y}\|^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{s} \\ t \end{bmatrix}, \end{aligned}$$

which is a BQP. Subsequently, SDR can be applied **[Tan-Rasmussen'01]**, **[Ma-Davidson-Wong-Luo-Ching'02]**.



Bit error rate performance under $(M_r, M_t) = (40, 40)$. 'ZF'— zero forcing; 'MMSE-DF'— min. mean square error with decision feedback; 'LRA'— lattice reduction aided. 'Randomization' will be explained shortly.



Complexity comparison of various MIMO detectors. SNR= 12dB. Sphere decoding is an exact ML method.

Additional Remarks about the MIMO Detection Application

- The idea is not restricted to spatial multiplexing! It can also be used in multiuser CDMA, space-time/freq./time-freq. coding, multiuser MIMO, and even blind MIMO **[Li-Bai-Ding'03]**, **[Ma-Vo-Davidson-Ching'06]**,...

- Extensions that have been considered:
  - MPSK constellations **[Ma-Ching-Ding'04]**;
  - higher-order QAM constellations **[Ma-Su-Jaldén-Chang-Chi'09]** (and refs. therein);
  - soft-in-soft-out MIMO detection (a.k.a. BICM-MIMO) **[Steingrimsson-Luo-Wong'03]**;
  - fast implementations **[Kisialiou-Luo-Luo'09]**, **[Wai-Ma-So'11]**

- Performance analysis for SDR MIMO detection:
  - diversity analysis **[Jaldén-Ottersten'08]**
  - probabilistic approximation accuracy analysis **[Kisialiou-Luo'10]**, **[So'10]**.

Alternative Interpretation of SDR: Solving QCQP in Expectation

- We return to the SDR solution approximation issue. Recall

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{x} \trianglerighteq_i b_i, \quad i=1,\ldots,m. \end{aligned} \qquad \text{(QCQP)}$$

- Let $\boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{X})$ where $\boldsymbol{X}$ is the covariance. Consider a stochastic QCQP:

$$\begin{aligned} \min_{\boldsymbol{X}\in\mathbb{S}^n, \; \boldsymbol{X}\succeq\mathbf{0}} \quad & \mathrm{E}_{\boldsymbol{\xi}\sim\mathcal{N}(\mathbf{0},\boldsymbol{X})}\{\boldsymbol{\xi}^T\boldsymbol{C}\boldsymbol{\xi}\} \\ \text{s.t.} \quad & \mathrm{E}_{\boldsymbol{\xi}\sim\mathcal{N}(\mathbf{0},\boldsymbol{X})}\{\boldsymbol{\xi}^T\boldsymbol{A}_i\boldsymbol{\xi}\} \trianglerighteq_i b_i, \quad i=1,\ldots,m, \end{aligned} \qquad \text{(E-QCQP)}$$

  where we manipulate the statistics of $\boldsymbol{\xi}$ so that the objective function is minimized & constraints are satisfied *in expectation*.

- One can show that (E-QCQP) is the same as the SDR

$$\begin{aligned} \min \quad & \mathrm{Tr}(\boldsymbol{C}\boldsymbol{X}) \\ \text{s.t.} \quad & \boldsymbol{X}\succeq\mathbf{0}, \quad \mathrm{Tr}(\boldsymbol{A}_i\boldsymbol{X}) \trianglerighteq_i b_i, \quad i=1,\ldots,m. \end{aligned} \qquad \text{(SDR)}$$

- The stochastic QCQP interpretation of SDR

$$\begin{aligned} \min_{\boldsymbol{X} \in \mathbb{S}^n} \quad & \mathrm{E}_{\boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{X})}\{\boldsymbol{\xi}^T \boldsymbol{C} \boldsymbol{\xi}\} \\ \text{s.t.} \quad & \mathrm{E}_{\boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{X})}\{\boldsymbol{\xi}^T \boldsymbol{A}_i \boldsymbol{\xi}\} \trianglerighteq_i b_i, \quad i = 1, \ldots, m \end{aligned} \qquad \text{(E-QCQP)}$$

  essentially sheds lights into a different way of approximating QCQP.

- What we could do is the following: generate a random vector $\boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{X}^\star)$ ($\boldsymbol{X}^\star$ is an SDR soln.), and modify $\boldsymbol{\xi}$ so that it is QCQP-feasible.

- Such a randomized QCQP soln. approx. may be performed multiple times, to get a better approx.

- (Believe it or not) The stochastic QCQP interpretation is the intuition behind many important theoretical SDR approx. accuracy results, including the famous Goemans-Williamson result **[Goemans-Williamson'95]**.

Example: Randomization in BQP or MIMO Detection

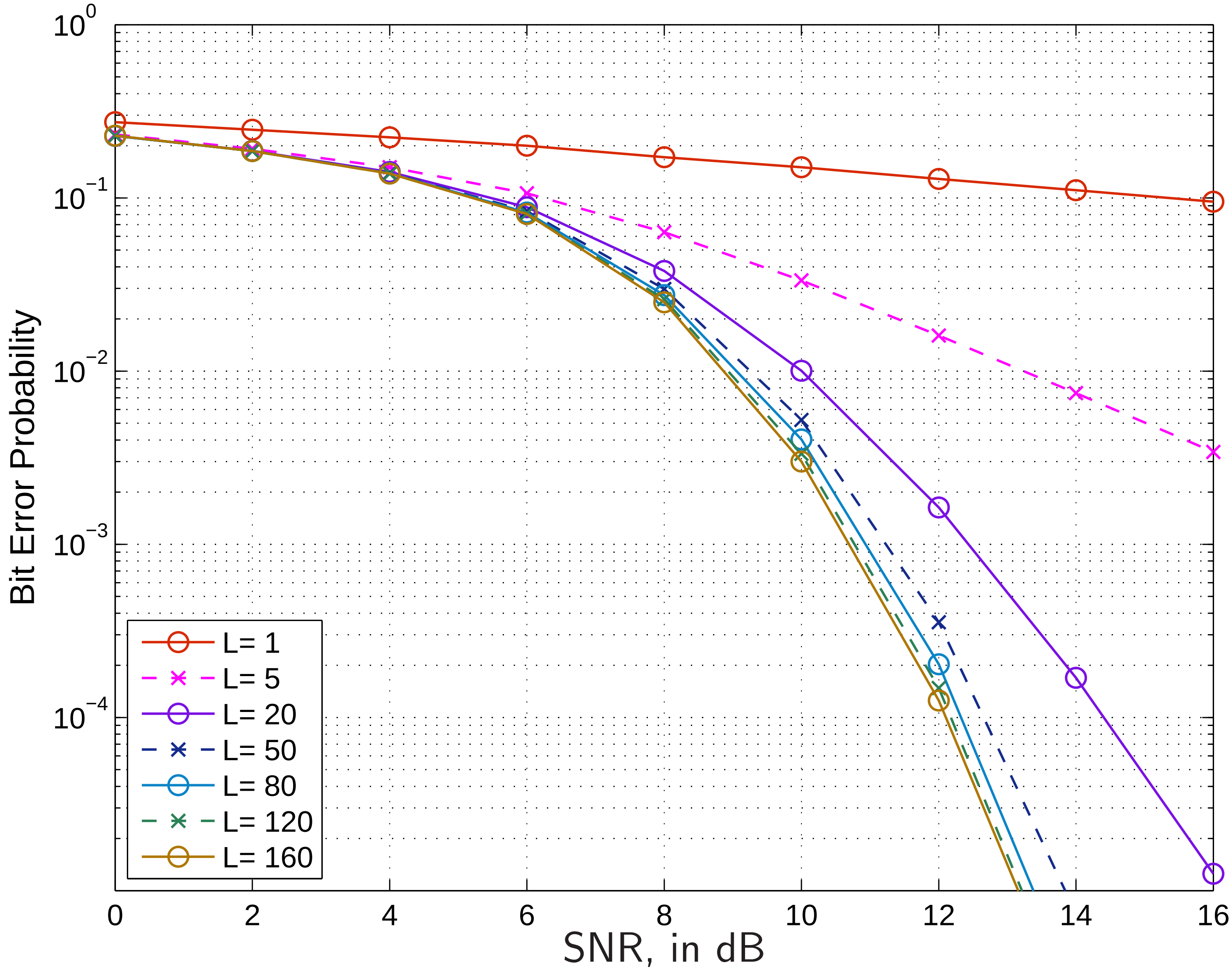
A simple (and very important) example for illustrating randomizations is BQP:

$$\begin{aligned} \min \quad & \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} \\ \text{s.t.} \quad & x_i^2 = 1, \quad i = 1, \ldots, n. \end{aligned} \tag{BQP}$$

Box 1. ***Gaussian Randomization Procedure for BQP***
given an SDR solution $\boldsymbol{X}^\star$, and a number of randomizations L.
for $\ell = 1, \ldots, L$
    generate $\boldsymbol{\xi}_\ell \sim \mathcal{N}(\mathbf{0}, \boldsymbol{X}^\star)$, and construct a feasible point

$$\tilde{\boldsymbol{x}}_\ell = \operatorname{sgn}(\boldsymbol{\xi}_\ell).$$

end
determine $\ell^\star = \arg \min_{\ell=1,\ldots,L} \tilde{\boldsymbol{x}}_\ell^T \boldsymbol{C} \tilde{\boldsymbol{x}}_\ell$.
output $\hat{\boldsymbol{x}} = \tilde{\boldsymbol{x}}_{\ell^\star}$ as an approximate solution to (BQP).



Performance of various no. of randomizations in MIMO detection. $M_t = M_r = 40$.

Complex-valued QCQP and SDR

- Consider a general complex-valued QCQP

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{C}^n} \quad & \boldsymbol{x}^H \boldsymbol{C} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^H \boldsymbol{A}_i \boldsymbol{x} \trianglerighteq_i b_i, \quad i = 1, \ldots, m, \end{aligned} \tag{1}$$

  where $\boldsymbol{C}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathbb{H}^n$; $\mathbb{H}^n$ denotes the set of $n \times n$ Hermitian matrices.

- Using the same idea, SDR can be derived for complex-valued QCQP:

$$\begin{aligned} \min_{\boldsymbol{X}\in\mathbb{H}^n} \quad & \mathrm{Tr}(\boldsymbol{C}\boldsymbol{X}) \\ \text{s.t.} \quad & \boldsymbol{X} \succeq \boldsymbol{0}, \quad \mathrm{Tr}(\boldsymbol{A}_i \boldsymbol{X}) \trianglerighteq_i b_i, \quad i = 1, \ldots, m. \end{aligned}$$

  The only difference is that the problem domain now is $\mathbb{H}^n$ (change '`symmetric`' to '`hermitian`' in your `CVX` code).

- Note that while the ideas leading to real and complex SDRs are the same, their performance may be different (we will see this later).

Application: Multicast Transmit Beamforming

Scenario: Common information broadcast in multiuser MISO downlink, assuming channel state information at the transmitter (CSIT).

- The transmit signal:

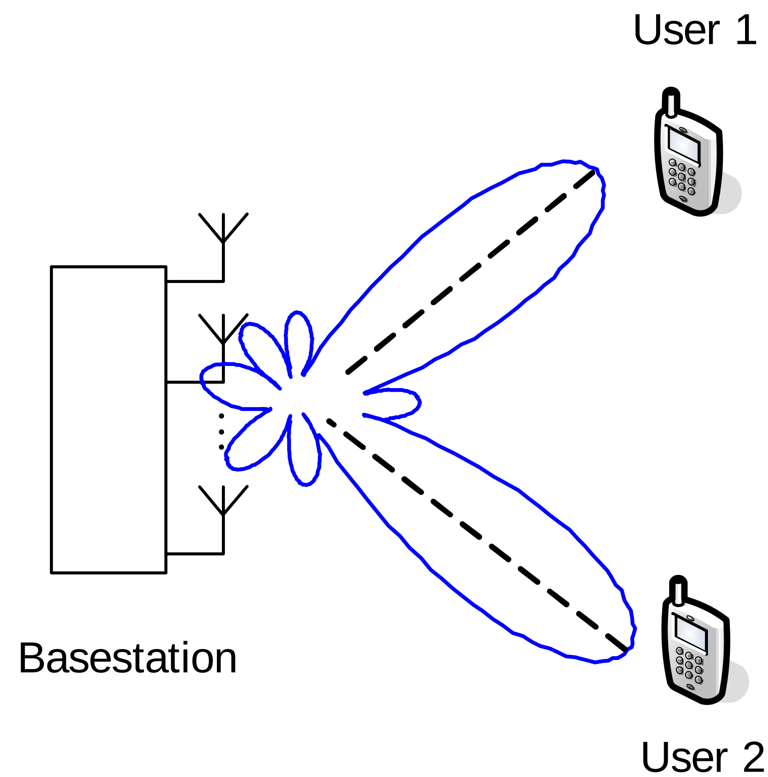
$$\boldsymbol{x}(t) = \boldsymbol{w}s(t),$$

where $s(t) \in \mathbb{C}$ is the tx. data stream, & $\boldsymbol{w} \in \mathbb{C}^{N_t}$ is the tx. beamvector.

- Received signal for user i:

$$y_i(t) = \boldsymbol{h}_i^H \boldsymbol{x}(t) + v_i(t),$$

where $\boldsymbol{h}_i \in \mathbb{C}^{N_t}$ is the channel of user i, & $v_i(t)$ is noise with variance σ_i^2.



- Consider a QoS-assured design:

$$\begin{aligned} \min_{\boldsymbol{w}\in\mathbb{C}^{N_t}} \quad & \|\boldsymbol{w}\|^2 \\ \text{s.t.} \quad & \textsf{SNR}_i \geq \gamma_i, \quad i = 1,\ldots,K, \end{aligned}$$

where each γ_i is a prescribed SNR requirement for user i, and

$$\textsf{SNR}_i = \mathrm{E}\{|\boldsymbol{h}_i^H \boldsymbol{w} s(t)|^2\}/\sigma_i^2 = \boldsymbol{w}^H \boldsymbol{R}_i \boldsymbol{w}/\sigma_i^2,$$

$$\boldsymbol{R}_i = \begin{cases} \boldsymbol{h}_i\boldsymbol{h}_i^H, & \boldsymbol{h}_i \text{ is available (instant CSIT)}, \\ \mathrm{E}\{\boldsymbol{h}_i\boldsymbol{h}_i^H\}, & \boldsymbol{h}_i \text{ is random with known 2nd order stat. (stat. CSIT)}. \end{cases}$$

- The design problem can be rewritten as a complex-valued QCQP

$$\begin{aligned} \min \quad & \|\boldsymbol{w}\|^2 \\ \text{s.t.} \quad & \boldsymbol{w}^H \boldsymbol{A}_i \boldsymbol{w} \geq 1, \quad i = 1,\ldots,K, \end{aligned}$$

where $\boldsymbol{A}_i = \boldsymbol{R}_i/\gamma_i\sigma_i^2$.

- This multicast problem is NP-hard in general, but can be approximated by SDR **[Sidiropoulos-Davidson-Luo'06]**.

A Randomization Example Relevant to Multicast Beamforming

Consider the problem

$$\begin{array}{rl} \min & \boldsymbol{x}^H \boldsymbol{C} \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x}^H \boldsymbol{A}_i \boldsymbol{x} \geq 1, \quad i = 1, \ldots, m, \end{array} \tag{$\dagger$}$$

where $\boldsymbol{C}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \succeq \boldsymbol{0}$.

Box 2. ***Gaussian Randomization Procedure for (†)***

given an SDR solution $\boldsymbol{X}^\star$, and a number of randomizations L.

for $\ell = 1, \ldots, L$

generate $\boldsymbol{\xi}_\ell \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{X}^\star)$, and construct a feasible point

$$\tilde{\boldsymbol{x}}_\ell = \frac{\boldsymbol{\xi}_\ell}{\sqrt{\min_{i=1,\ldots,m} \boldsymbol{\xi}_\ell^H \boldsymbol{A}_i \boldsymbol{\xi}_\ell}}$$

end

determine $\ell^\star = \arg \min_{\ell=1,\ldots,L} \tilde{\boldsymbol{x}}_\ell^H \boldsymbol{C} \tilde{\boldsymbol{x}}_\ell$.

output $\hat{\boldsymbol{x}} = \tilde{\boldsymbol{x}}_{\ell^\star}$ as an approximate solution to (†).

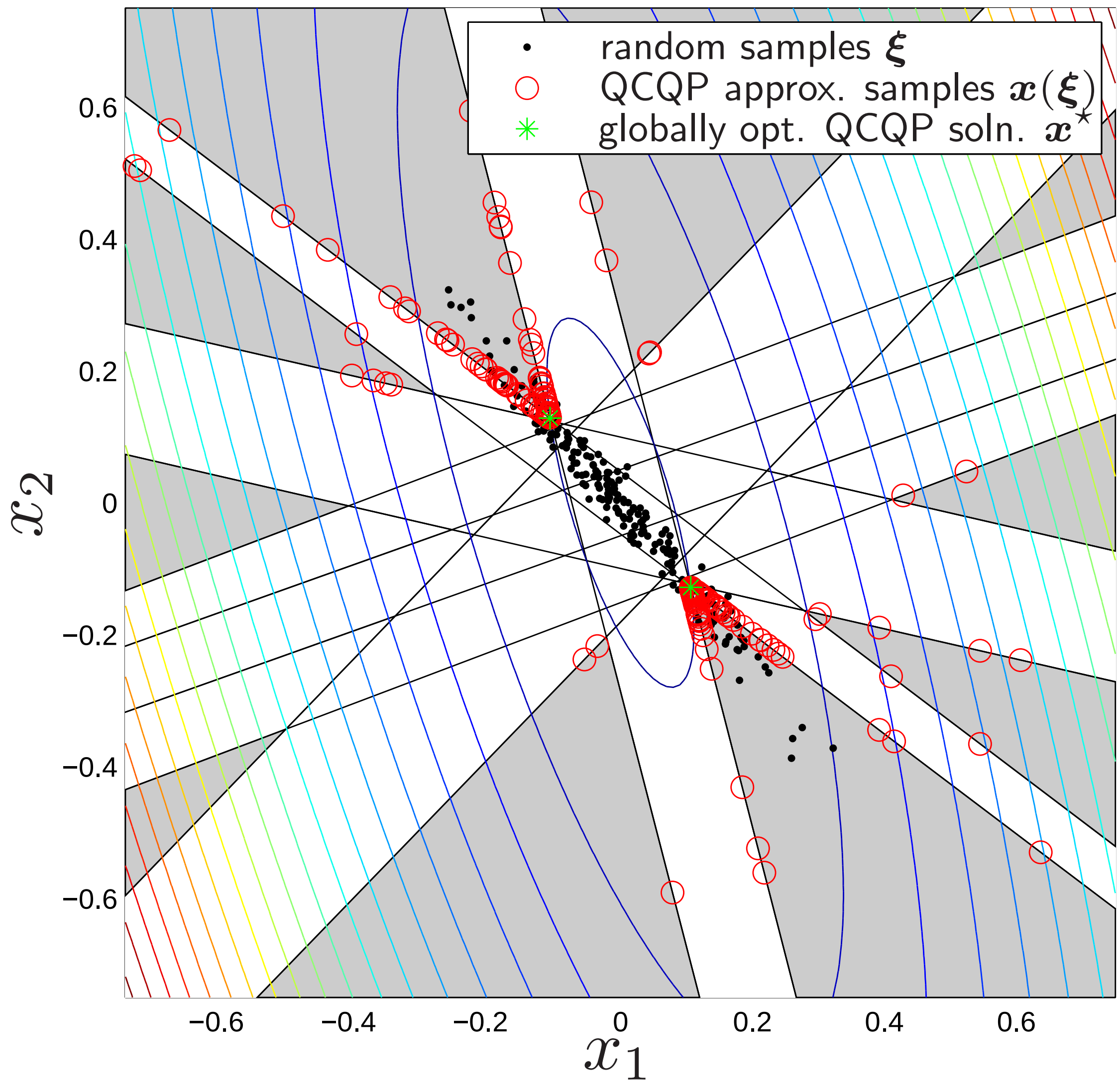
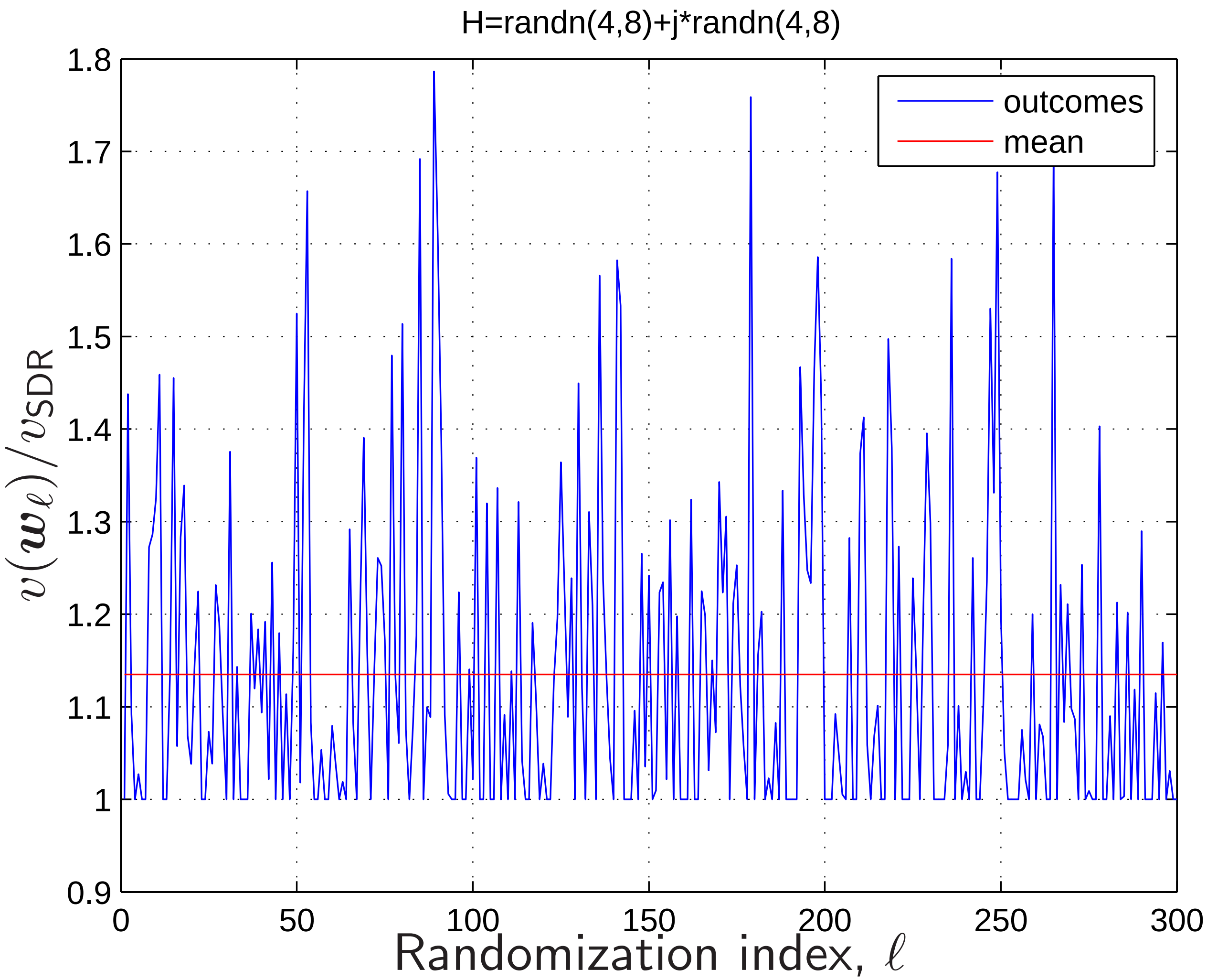


Illustration of randomizations in $\mathbb{R}^2$, for Problem (†). The gray area is the feasible set and colored lines the contour of the objective.



Approximation accuracy of Gaussian randomization in multicast beamforming. $N_t = 4$, $K = 8$, $v(\boldsymbol{w}) = \|\boldsymbol{w}\|^2$ is the objective value, $v_{\sf SDR}$ is the optimal value of SDR. Note that for any feasible $\boldsymbol{w}$, $v(\boldsymbol{w})/v_{\sf SDR} \geq v_{\sf QP}/v_{\sf SDR}$ where $v_{\sf QP}$ is the optimal value of QCQP. Courtesy to T.-H. Chang and Z.-Q. Luo.

Extension to Complex-Valued Separable QCQP

- Consider a further extension, called complex-valued separable QCQP:

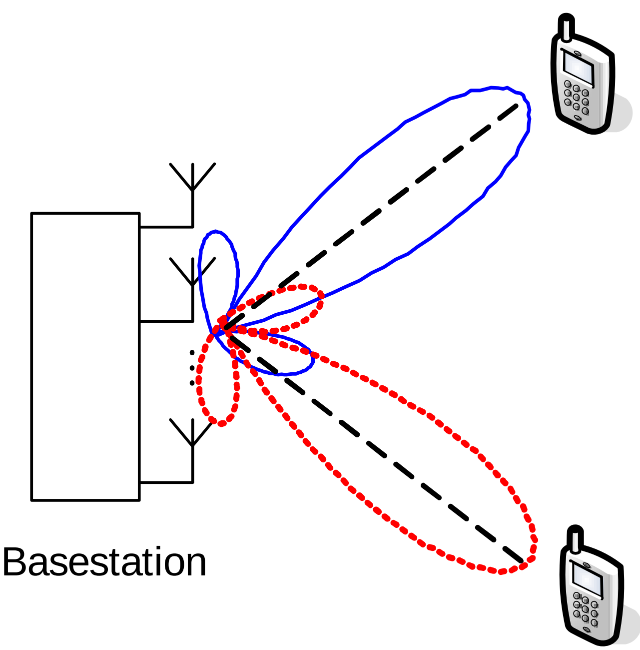
$$\begin{aligned}\min_{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\in\mathbb{C}^n} \quad & \textstyle\sum_{i=1}^k \boldsymbol{x}_i^H \boldsymbol{C}_i \boldsymbol{x}_i \\ \text{s.t.} \quad & \textstyle\sum_{l=1}^k \boldsymbol{x}_l^H \boldsymbol{A}_{i,l} \boldsymbol{x}_l \trianglerighteq_i b_i, \quad i=1,\ldots,m.\end{aligned}$$

- By writing $\boldsymbol{X}_i = \boldsymbol{x}_i\boldsymbol{x}_i^H$ for all i, and then "semidefinite-relaxing" them, we obtain an SDR

$$\begin{aligned}\min_{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_k\in\mathbb{H}^n} \quad & \textstyle\sum_{i=1}^k \mathrm{Tr}(\boldsymbol{C}_i\boldsymbol{X}_i) \\ \text{s.t.} \quad & \textstyle\sum_{l=1}^k \mathrm{Tr}(\boldsymbol{A}_{i,l}\boldsymbol{X}_i) \trianglerighteq_i b_i, \quad i=1,\ldots,m, \\ & \boldsymbol{X}_1 \succeq \boldsymbol{0},\ldots,\boldsymbol{X}_k \succeq \boldsymbol{0}.\end{aligned}$$

Application: Unicast Transmit Downlink Beamforming

Scenario: multiuser MISO downlink; each user receives an individual data stream.



- Transmit signal:

$$\boldsymbol{x}(t) = \sum_{i=1}^{K} \boldsymbol{w}_i s_i(t),$$

where $s_i(t) \in \mathbb{C}$ is the data stream for user i, & $\boldsymbol{w}_i \in \mathbb{C}^{N_t}$ its tx. beamvector.

- Received signal of user i:

$$\begin{aligned} y_i(t) &= \boldsymbol{h}_i^H \boldsymbol{x}(t) + v_i(t) \\ &= \boldsymbol{h}_i^H \boldsymbol{w}_i s_i(t) + \underbrace{\sum_{l \neq i} \boldsymbol{h}_i^H \boldsymbol{w}_l s_l(t)}_{\text{interference}} + v_i(t). \end{aligned}$$

- The signal-to-interference-and-noise ratio (SINR) of user i:

$$\text{SINR}_i = \frac{\boldsymbol{w}_i^H \boldsymbol{R}_i \boldsymbol{w}_i}{\sum_{l \neq i} \boldsymbol{w}_l^H \boldsymbol{R}_i \boldsymbol{w}_l + \sigma_i^2},$$

where $\boldsymbol{R}_i = \boldsymbol{h}_i \boldsymbol{h}_i^H$ for instant. CSIT, and $\boldsymbol{R}_i = \mathrm{E}\{\boldsymbol{h}_i \boldsymbol{h}_i^H\}$ for stat. CSIT.

- Consider the QoS-assured design:

$$\begin{aligned} \min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_K \in \mathbb{C}^{N_t}} \quad & \textstyle\sum_{i=1}^K \|\boldsymbol{w}_i\|^2 \\ \text{s.t.} \quad & \frac{\boldsymbol{w}_i^H \boldsymbol{R}_i \boldsymbol{w}_i}{\sum_{l \neq i} \boldsymbol{w}_l^H \boldsymbol{R}_i \boldsymbol{w}_l + \sigma_i^2} \geq \gamma_i, \quad i = 1, \ldots, K \end{aligned} \qquad (\dagger)$$

and its SDR

$$\begin{aligned} \min_{\boldsymbol{W}_1,\ldots,\boldsymbol{W}_K \in \mathbb{H}^{N_t}} \quad & \textstyle\sum_{i=1}^K \mathrm{Tr}(\boldsymbol{W}_i) \\ \text{s.t.} \quad & \mathrm{Tr}(\boldsymbol{R}_i \boldsymbol{W}_i) \geq \gamma_i (\textstyle\sum_{l \neq i} \mathrm{Tr}(\boldsymbol{R}_i \boldsymbol{W}_l) + \sigma_i^2), \ i = 1, \ldots, K, \\ & \boldsymbol{W}_1, \ldots, \boldsymbol{W}_K \succeq \boldsymbol{0}. \end{aligned} \qquad (\ddagger)$$

- (‡) is shown to have a rank-one solution for $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_K \succeq \boldsymbol{0}$, via uplink-downlink duality **[Bengtsson-Ottersten'01]**; SDR is optimal to (†), so to speak!

- We will introduce an "easy" way to identify rank-one SDR instances.

SDR Versus Nonlinear Programming: They complement, not compete

- Since SDR is an approximation method, as an alternative one may choose to approximate (QCQP) by a nonlinear programming method (NPM) (like, SQP in the MATLAB Optimization Toolbox).

- So should we compare SDR and NPM?

- The interesting argument is that they complement each other, instead of competing:
  - An NPM depends much on a 'good' starting point, and that's usually the missing piece.
  - To SDR, NPMs may serve as a local refinement of the solution.

- One may consider a two-stage approach where SDR is used as a starting point for NPMs.

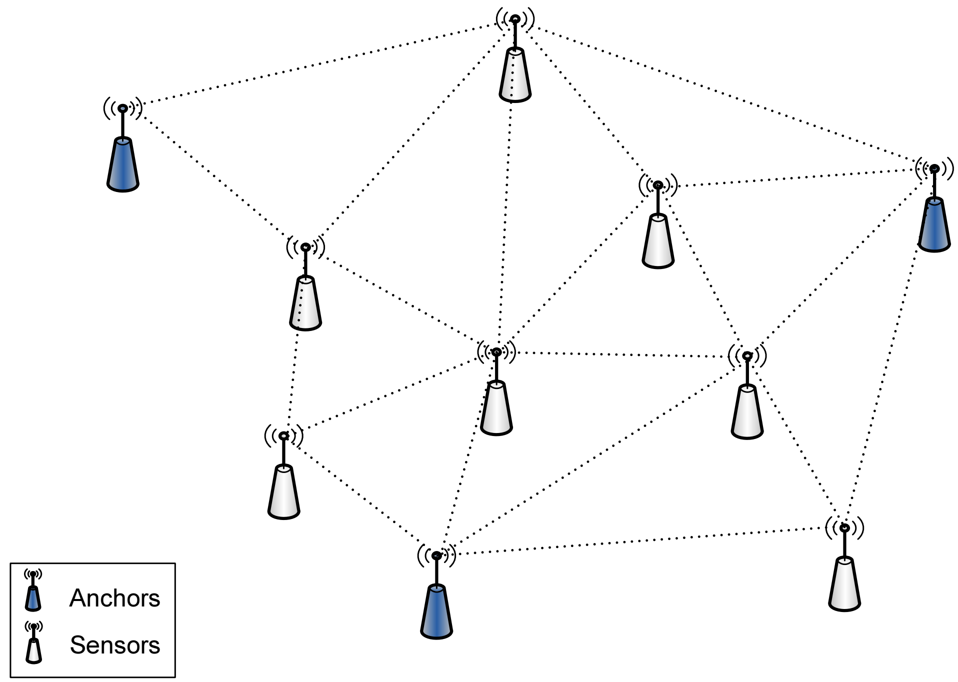
Application: Sensor Network Localization

The sensor network localization (SNL) problem is to determine the (x, y) coordinates of the sensors, given distance information between sensors.

- In ad-hoc sensor networks, the sensor locations may not be known.

- A sensor may acquire its location by equipping it with GPS, but this may be too expensive.

- We may have several *anchor* sensors that have self-localization capability, though.



- Since sensors can communicate with each other, each sensor pair can work out their distance (e.g., by measuring the time-of-arrival info., or by ping-pong).

- The inter-sensor distances, together with anchor locations, can be used to estimate all the sensor locations in a joint fashion.

- Let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$, $\boldsymbol{x}_i \in \mathbb{R}^2$ for all i, be the collection of all (unknown) sensor coordinates.
- Let $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$, $\boldsymbol{a}_i \in \mathbb{R}^2$, be the collection of all (known) anchor coordinates.
- The distance between sensor i and sensor j is

$$d_{ij} = \sqrt{(x_{i,1} - x_{j,1})^2 + (x_{i,2} - x_{j,2})^2} = \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$$

  Likewise, the distance between sensor i and anchor j is

$$\bar{d}_{ij} = \|\boldsymbol{x}_i - \boldsymbol{a}_j\|$$

  The obtained d_{ij} & $\bar{d}_{ij}$ are assumed noiseless (extension for noisy cases available).

- The SNL problem here is that of finding $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ such that

$$\begin{aligned} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2 &= d_{ij}^2, \quad (i,j) \in E_{ss} \\ \|\boldsymbol{x}_i - \boldsymbol{a}_j\|^2 &= \bar{d}_{ij}^2, \quad (i,j) \in E_{sa} \end{aligned}$$

  where E_{ss} & E_{sa} are the sensor-to-sensor & sensor-to-anchor edge sets, resp.

- Let $\boldsymbol{X} = [\ \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\ ] \in \mathbb{R}^{2\times n}$. The SNL problem is written as

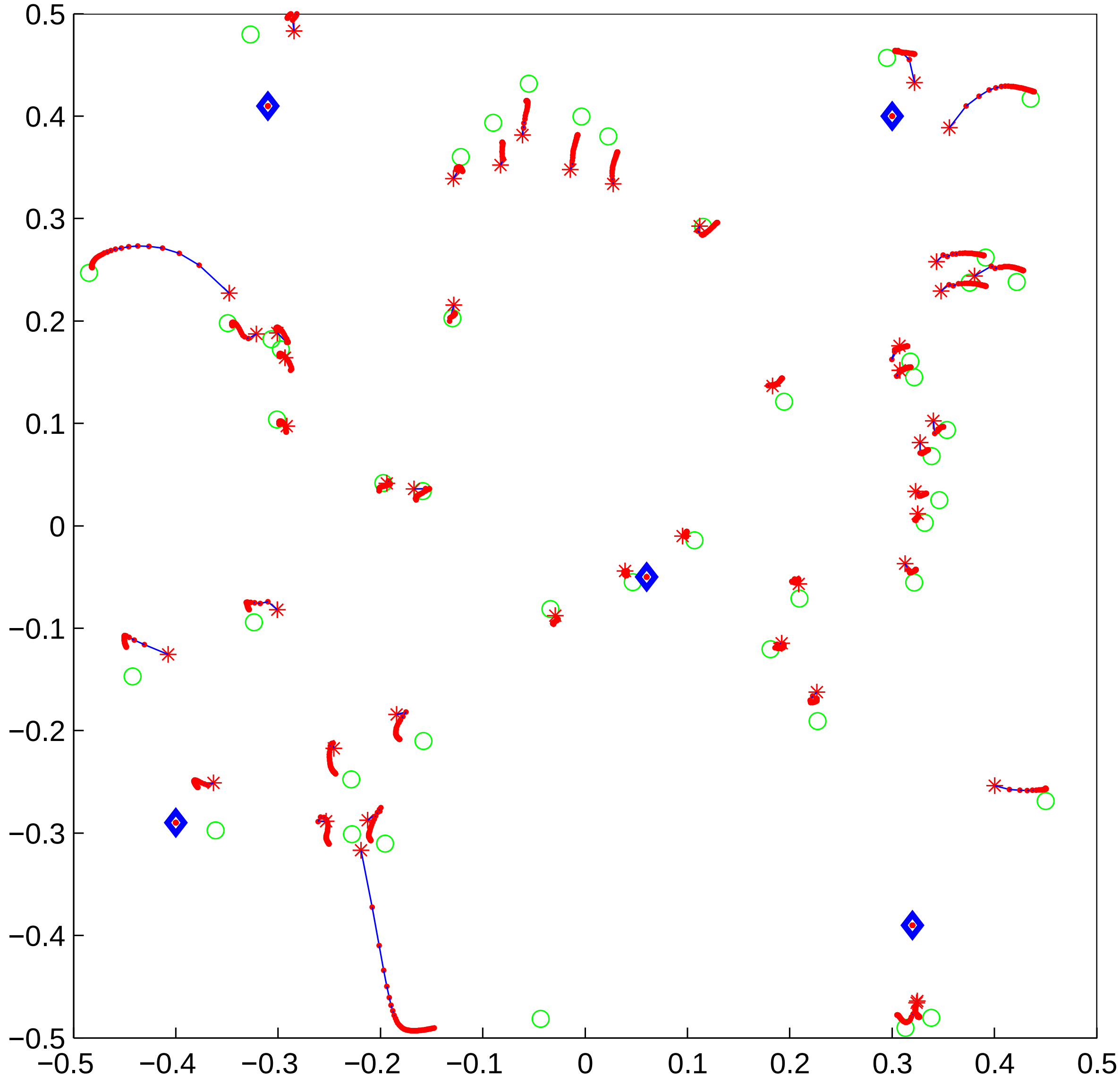
$$
\begin{aligned}
\text{find} \quad & \boldsymbol{X} \\
\text{s.t.} \quad & \boldsymbol{x}_i^T \boldsymbol{x}_i - 2\boldsymbol{x}_i^T \boldsymbol{x}_j + \boldsymbol{x}_j^T \boldsymbol{x}_j = d_{ij}^2, \quad (i,j) \in E_{ss} \\
& \boldsymbol{x}_i^T \boldsymbol{x}_i - 2\boldsymbol{x}_i^T \boldsymbol{a}_j + \boldsymbol{a}_j^T \boldsymbol{a}_j = \bar{d}_{ij}^2, \quad (i,j) \in E_{sa}
\end{aligned}
$$

- Let $\boldsymbol{Y} = \boldsymbol{X}^T \boldsymbol{X} \in \mathbb{R}^{n \times n}$. The SNL problem have an equivalent formulation

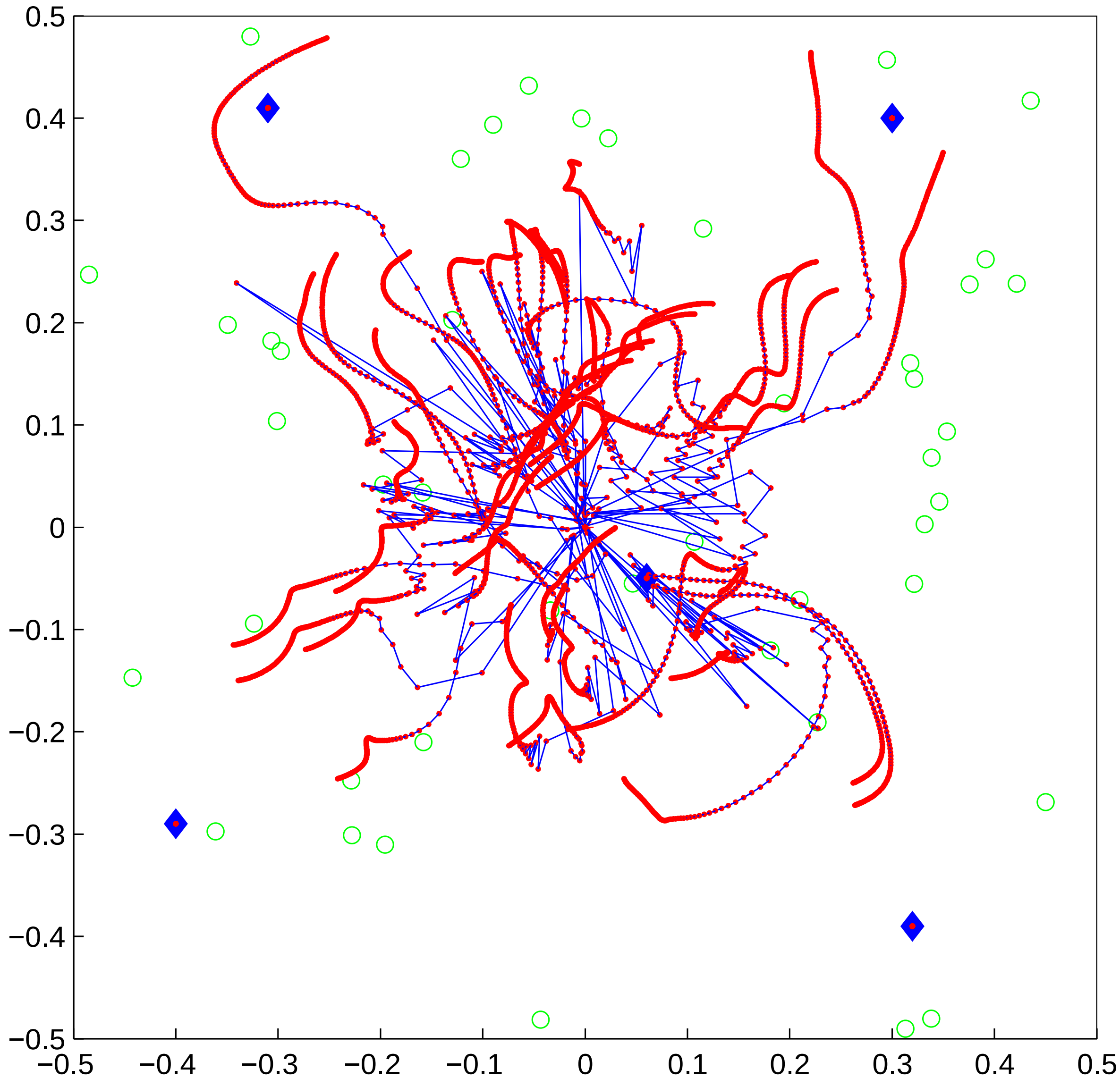
$$
\begin{aligned}
\text{find} \quad & \boldsymbol{X}, \boldsymbol{Y} \\
\text{s.t.} \quad & Y_{ii} - 2Y_{ij} + Y_{jj} = d_{ij}^2, \quad (i,j) \in E_{ss} \\
& Y_{ii} - 2\boldsymbol{x}_i^T \boldsymbol{a}_j + \boldsymbol{a}_j^T \boldsymbol{a}_j = \bar{d}_{ij}^2, \quad (i,j) \in E_{sa} \\
& \boldsymbol{Y} = \boldsymbol{X}^T \boldsymbol{X}
\end{aligned}
$$

- Naturally (and after you have seen how SDR operates), the SNL problem can be approximated by SDR:

$$
\begin{aligned}
\text{find} \quad & \boldsymbol{X}, \boldsymbol{Y} \\
\text{s.t.} \quad & Y_{ii} - 2Y_{ij} + Y_{jj} = d_{ij}^2, \quad (i,j) \in E_{ss} \\
& Y_{ii} - 2\boldsymbol{x}_i^T \boldsymbol{a}_j + \boldsymbol{a}_j^T \boldsymbol{a}_j = \bar{d}_{ij}^2, \quad (i,j) \in E_{sa} \\
& \boldsymbol{Y} \succeq \boldsymbol{X}^T \boldsymbol{X}
\end{aligned}
$$



SDR (ML-SNL formulation), plus a 2nd-stage solution refinement by gradient descent. The distance measurements are noisy. ○: true sensor locations; ◊: anchor locations; ∗: SDR solution; —— : gradient descent trajectory (50 iterations).



Gradient descent ML-SNL with a random starting point. ○: true sensor locations; ◊: anchor locations; — : gradient descent trajectory (50 iterations).

Part II: Theory

Provable Approximation Accuracies

The following problem has been of great interest to optimization theorists, and it has enormous implications in practice.

- Let $v(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x}$, and denote the optimal values of (QCQP) and (SDR) by

$$\begin{aligned} v_{\mathsf{QP}} = \min \quad & \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} \trianglerighteq_i b_i, \quad i = 1, \ldots, m \\ v_{\mathsf{SDR}} = \min \quad & \mathrm{Tr}(\boldsymbol{C}\boldsymbol{X}) \\ \text{s.t.} \quad & \boldsymbol{X} \succeq \boldsymbol{0}, \quad \mathrm{Tr}(\boldsymbol{A}_i \boldsymbol{X}) \trianglerighteq_i b_i, \quad i = 1, \ldots, m \end{aligned}$$

  Moreover, let $\hat{\boldsymbol{x}}$ be an approximate solution of (QCQP), say, using randomization. Note that

$$v_{\mathsf{QP}} \le v(\hat{\boldsymbol{x}}).$$

- The problem is to prove a constant γ such that

$$v(\hat{\boldsymbol{x}}) \le \gamma v_{\mathsf{QP}}$$

  in a worst case sense, or with high probability. A γ close to 1 would mean a near-optimal accuracy.

The Seminal Approx. Accuracy Result by Goemans & Williamson

- Consider
$$\begin{aligned} v_{\sf QP} = \max_{\boldsymbol{x} \in \mathbb{R}^n} \quad & \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} \\ \text{s.t.} \quad & x_i^2 = 1, \quad i = 1, \ldots, n \end{aligned}$$
with $\boldsymbol{C} \succeq \boldsymbol{0}$, $C_{ij} \leq 0$ for all $i \neq j$ (the so-called MAXCUT in network optimization).

- In **[Goemans-Williamson'95]**, it was shown that if the randomization procedure is used, then
$$\gamma v_{\sf QP} \leq \mathrm{E}\{v(\hat{\boldsymbol{x}})\} \leq v_{\sf QP}$$
where $\gamma \approx 0.87856$.

- The work by Goeman and Williamson has triggered much interest, resulting in many more approx. accuracy results being established for a wider class of problems.

Approx. Accuracy Result for Quadratic Minimization

- Consider now the problem

$$\begin{aligned} v_{\mathsf{QP}} = \min_{\boldsymbol{x} \in \mathbb{C}^n} \quad & \boldsymbol{x}^H \boldsymbol{C} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^H \boldsymbol{A}_i \boldsymbol{x} \geq 1, \quad i = 1, \ldots, m \end{aligned} \tag{$\dagger$}$$

  for $\boldsymbol{C}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \succeq \mathbf{0}$, which arises in multicast downlink beamforming.

- It was shown in **[Luo-Sidiropoulos-Tseng-Zhang'07]** that if the randomization procedure in Box 2 is used, then with high probability (instead of just in expectation),

$$v_{\mathsf{QP}} \leq v(\hat{\boldsymbol{x}}) \leq \gamma v_{\mathsf{QP}},$$

  where $\gamma = 8m$.

  - In multicast beamforming, this result says that we can produce a transmit beamforming vector that satisfies all the prescribed SNR requirements and whose power is at most $8m$ times the optimal.
  - Notice that this ratio accommodates the worst possible problem instance $\{\boldsymbol{C}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m\}$. In practice, the approximation accuracies are usually much better— a phenomenon that deserves further investigation.

problem	approx. accuracy γ; see (21)-(22) for def.	references
Boolean QP $\max_{\boldsymbol{x}\in\mathbb{R}^n} \ \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x}$ s.t. $x_i^2 = 1, \ i = 1,\ldots,n$	$\gamma = \begin{cases} 0.87856, & \boldsymbol{C} \succeq \boldsymbol{0}, \ C_{ij} \leq 0 \ \forall i \neq j \\ 2/\pi \simeq 0.63661, & \boldsymbol{C} \succeq \boldsymbol{0} \\ 1 \text{ (opt.)}, & C_{ij} \geq 0, \forall i \neq j \end{cases}$	Goemans-Williamson [2], Nesterov [3], Zhang [6]. Relevant applications: [24]–[26]
Complex k-ary QP $\max_{\boldsymbol{x}\in\mathbb{C}^n} \ \boldsymbol{x}^H\boldsymbol{C}\boldsymbol{x}$ s.t. $x_i \in \{1,\omega,\ldots,\omega^{k-1}\},$ $i = 1,\ldots,n$ where $\omega = e^{j2\pi/k}$, and $k > 1$ is an integer.	For $\boldsymbol{C} \succeq \boldsymbol{0}$, $\gamma = \frac{(k\sin(\pi/k))^2}{4\pi}.$ e.g., $\gamma = 0.7458$ for $k = 8$, $\gamma = 0.7754$ for $k = 16$.	Zhang-Huang [7], So-Zhang-Ye [8]. Relevant applications: [27], [37]
Complex constant-modulus QP $\max_{\boldsymbol{x}\in\mathbb{C}^n} \ \boldsymbol{x}^H\boldsymbol{C}\boldsymbol{x}$ s.t. $\lvert x_i\rvert^2 = 1, \ i = 1,\ldots,n$	For $\boldsymbol{C} \succeq \boldsymbol{0}$, $\gamma = \pi/4 = 0.7854.$ Remark: coincide with complex k-ary QP as $k \to \infty$.	Zhang-Huang [7], So-Zhang-Ye [8].
$\max_{\boldsymbol{x}\in\mathbb{C}^n} \ \boldsymbol{x}^H\boldsymbol{C}\boldsymbol{x}$ s.t. $(\lvert x_1\rvert^2,\ldots,\lvert x_n\rvert^2) \in \mathcal{F}$ where $\mathcal{F} \subset \mathbb{R}^n$ is a closed convex set.	The same approx. ratio as in complex constant-modulus QP; i.e., $\gamma = \pi/4$ for $\boldsymbol{C} \succeq \boldsymbol{0}$. If the problem is reduced to the real-valued case, then the approx. ratio results are the same as that in Boolean QP.	Ye [4], Zhang [6].
$\max_{\boldsymbol{x}\in\mathbb{R}^n} \ \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x}$ s.t. $\boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{x} \leq 1, \ i = 1,\ldots,m$ where $\boldsymbol{A}_1,\ldots,\boldsymbol{A}_m \succeq \boldsymbol{0}$.	For any $\boldsymbol{C} \in \mathbb{S}^n$, $\gamma = \frac{1}{2\ln(2m\mu)}$ where $\mu = \min\{m, \max_i \operatorname{rank}(\boldsymbol{A}_i)\}$.	Nemirovski-Roos-Terlaky [5]. Extensions: Ye [72], Luo-Sidiropoulos-Tseng-Zhang [9] and So-Ye-Zhang [71].

Known approximation accuracies for quadratic maximization problems. The reference numbers refer to those in our Signal Processing Magazine article.

problem	approx. accuracy γ; see (18)-(19) for def.	references
$\min_{\boldsymbol{x}\in\mathbb{C}^n} \ \boldsymbol{x}^H\boldsymbol{C}\boldsymbol{x}$ s.t. $\boldsymbol{x}^H\boldsymbol{A}_i\boldsymbol{x} \geq 1, \ i=1,\ldots,m$ where $\boldsymbol{A}_1,\ldots,\boldsymbol{A}_m \succeq \boldsymbol{0}$.	$\gamma = 8m.$ If the problem is reduced to the real-valued case, then $\gamma = \frac{27m^2}{\pi}.$	Luo-Sidiropoulos-Tseng-Zhang [9]; see also So-Ye-Zhang [71]. Relevant applications: [29]
MIMO Detection $\min_{\boldsymbol{x}\in\mathbb{R}^n} \ \|\boldsymbol{y}-\boldsymbol{H}\boldsymbol{x}\|_2^2$ s.t. $x_i^2 = 1, \ i=1,\ldots,n$ where $\boldsymbol{y}=\boldsymbol{H}\boldsymbol{s}+\boldsymbol{v}$; $\boldsymbol{H}\in\mathbb{C}^{n\times n}$ has i.i.d. standard complex Gaussian entries; $s_i^2=1$ for $i=1,\ldots,n$; and $\boldsymbol{v}\in\mathbb{C}^n$ has i.i.d. complex mean zero Gaussian entries with variance σ^2.	For $\sigma^2 \geq 60n$ (which corresponds to the low signal-to-noise ratio (SNR) region), with probability at least $1-3\exp(-n/6)$, $\gamma \leq \frac{11}{2}.$ For $\sigma^2 = \mathcal{O}(1)$ (which corresponds to the high SNR region), with probability at least $1-\exp(-\mathcal{O}(n))$, $\gamma = 1,$ i.e. the SDR is tight.	Kisialiou-Luo [67], So [69]. Extensions: So [68], [69]. Related: Jaldén-Ottersten [66]. Relevant applications: [17]–[20], [22], [23]

Known approximation accuracies for quadratic minimization problems. The reference numbers refer to those in our Signal Processing Magazine article.

Rank Reduction in SDR

- Now you may notice that an SDR methodology basically has the following steps:
  1) formulate a hard problem (nonconvex QCQP) as a rank-one-constrained SDP
  2) remove the rank constraint to obtain an SDP
  3) use some methods, such as randomizations, to produce an approximate solution to the original problem.
- Apparently, the lower the rank of the SDP solution, the better the approximation we would expect.
- Unfortunately, we cannot guarantee a low rank solution for the SDP in general.
- But we can identify special cases where the SDP solution rank is low, and, sometimes, one.

Shapiro-Barvinok-Pataki (SBP) Result

- Consider the real-valued SDP (or SDR)

$$\begin{aligned} \min_{\boldsymbol{X} \in \mathbb{S}^n} \quad & \mathrm{Tr}(\boldsymbol{C}\boldsymbol{X}) \\ \text{s.t.} \quad & \boldsymbol{X} \succeq \mathbf{0}, \quad \mathrm{Tr}(\boldsymbol{A}_i \boldsymbol{X}) \trianglerighteq_i b_i, \quad i = 1, \ldots, m \end{aligned} \qquad \text{(SDR)}$$

SBP Result [Pataki'98]: there exists an optimal solution $\boldsymbol{X}^\star$ such that

$$\frac{\mathrm{rank}(\boldsymbol{X}^\star)(\mathrm{rank}(\boldsymbol{X}^\star) + 1)}{2} \leq m$$

- In particular, SBP result implies that for $m \leq 2$, a rank-1 $\boldsymbol{X}^\star$ exists. Hence,

> For a real-valued QCQP with $m \leq 2$, SDR is tight; i.e., solving the SDR is equivalent to solving the original QCQP.

- Note that a rank reduction algorithm may be required to turn an SDP solution to a rank-one solution **[Ye-Zhang'03]**.

Complex Extension of the Rank Reduction Result

- Let us consider the extension to the complex-valued SDP

$$\begin{aligned} \min_{\boldsymbol{X} \in \mathbb{H}^n} \quad & \text{Tr}(\boldsymbol{C}\boldsymbol{X}) \\ \text{s.t.} \quad & \boldsymbol{X} \succeq \boldsymbol{0}, \quad \text{Tr}(\boldsymbol{A}_i \boldsymbol{X}) \trianglerighteq_i b_i, \quad i = 1, \ldots, m \end{aligned}$$

- In this case, the SBP result can be generalized to **[Huang-Palomar'09]**

$$\text{rank}(\boldsymbol{X}^\star)^2 \leq m$$

  and the direct consequence is that

  > For a complex-valued QCQP with $m \leq 3$, SDR is tight; i.e., solving the SDR is equivalent to solving the original QCQP.

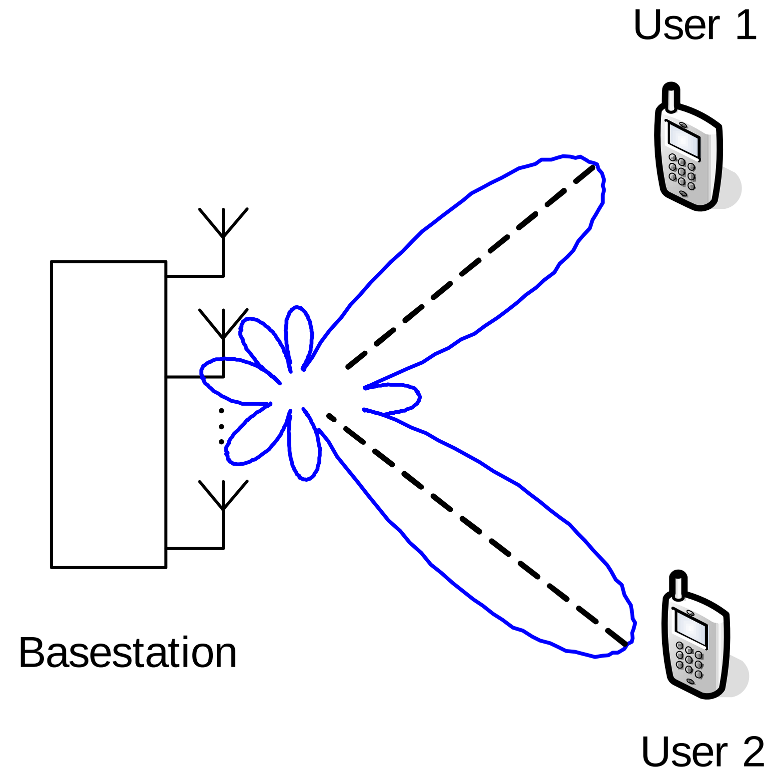
- A complex rank-1 decomposition algorithm for $m \leq 3$ is available **[Huang-Zhang'07]**.

Application Revisited: Multicast Beamforming

- Recall the multicast beamforming problem:

$$\begin{aligned} \min_{\boldsymbol{w} \in \mathbb{C}^{N_t}} \quad & \|\boldsymbol{w}\|^2 \\ \text{s.t.} \quad & \mathsf{SNR}_i = \frac{\boldsymbol{w}\boldsymbol{R}_i\boldsymbol{w}}{\sigma_i^2} \geq \gamma_i, \\ & i = 1, \ldots, K, \end{aligned}$$

K being the number of users.



- By the SBP result, SDR solves the multicast problem optimally for $K \leq 3$.

Further Extension of the Rank Reduction Result

- Recall the problem

$$\begin{aligned} \min_{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_k \in \mathbb{H}^n} \quad & \textstyle\sum_{i=1}^k \mathrm{Tr}(\boldsymbol{C}_i \boldsymbol{X}_i) \\ \text{s.t.} \quad & \textstyle\sum_{l=1}^k \mathrm{Tr}(\boldsymbol{A}_{i,l} \boldsymbol{X}_i) \trianglerighteq_i b_i, \quad i = 1, \ldots, m, \\ & \boldsymbol{X}_1 \succeq \mathbf{0}, \ldots, \boldsymbol{X}_k \succeq \mathbf{0}, \end{aligned}$$

which is an SDR of the so-called separable QCQP.

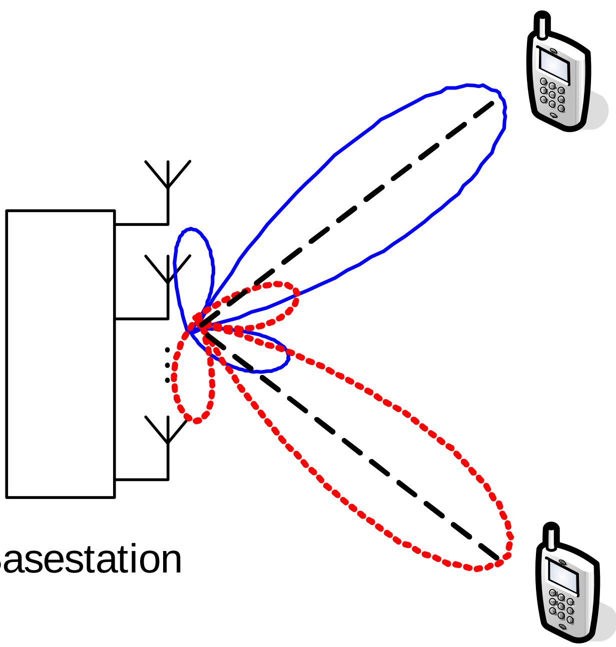
- A generalization of the SBP result **[Huang-Palomar'09]**:

$$\textstyle\sum_{i=1}^k \mathrm{rank}(\boldsymbol{X}_i^\star)^2 \leq m,$$

and, as a subsequent result:

> Suppose that an SDR solution cannot have $\boldsymbol{X}_i^\star = \mathbf{0}$ for any i. Then SDR is tight for $m \leq k + 2$.

Application Revisited: Unicast Beamforming



- Recall the design problem

$$\begin{aligned} \min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_K \in \mathbb{C}^{N_t}} \quad & \sum_{i=1}^K \|\boldsymbol{w}_i\|^2 \\ \text{s.t.} \quad & \frac{\boldsymbol{w}_i^H \boldsymbol{R}_i \boldsymbol{w}_i}{\sum_{l \neq i} \boldsymbol{w}_l^H \boldsymbol{R}_i \boldsymbol{w}_l + \sigma_i^2} \geq \gamma_i, \\ & i = 1, \ldots, K \end{aligned} \qquad (\dagger)$$

which is a separable QCQP with K variables (beamvectors) and K constraints (SINR req.).

- By the SBP rank reduction result, SDR solves (†) optimally for any $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_K$, regardless of $\boldsymbol{R}_i \succeq \boldsymbol{0}$ or not.

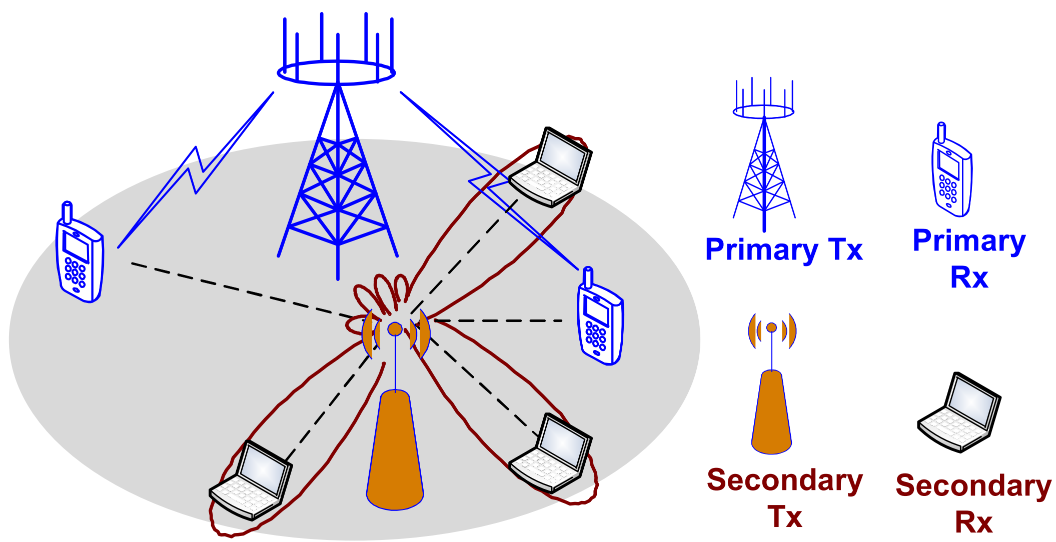
- And hey, it's still fine if you put two more quadratic constraints in (†)!

Cognitive Radio (CR) Beamforming: A Further Example

- Goal: access the channel owned by primary users (PUs) through spectrum sharing.

- Idea: the CR system avoids excessive interference to the PUs through tx. opt.

- **Scenario:** MISO downlink with the CR (or secondary) system, either unicast or multicast; K secondary users (SUs); L single-antenna PUs



- Assume known CSIT from the secondary transmitter to the PUs.

- Consider the multicast case.
  - tx. and rx. model for SUs: same as the previous multicast model.
  - Interference to the lth PU:

$$|\boldsymbol{g}_l^H \boldsymbol{w}|^2$$

  where $\boldsymbol{g}_l$ is the channel from the secondary transmitter to the lth PU.

- Design problem **[Phan-Vorobyov-Sidiropoulos-Tellambura'09]**:

$$\begin{aligned} \min_{\boldsymbol{w}} \ & \|\boldsymbol{w}\|^2 \\ \text{s.t.} \ & \mathsf{SNR}_{\mathsf{SU},i} = \boldsymbol{w}^H \boldsymbol{R}_k \boldsymbol{w}/\sigma_k^2 \geq \gamma_k, \ k = 1, \ldots, K, \\ & \boldsymbol{w}^H \boldsymbol{G}_l \boldsymbol{w} \leq \delta_l, \ l = 1, \ldots, L \quad \text{(interference temperature (IT) constraints)} \end{aligned}$$

  where $\boldsymbol{G}_l$ is the CSIT of lth PU (defined in the same way as $\boldsymbol{R}_k$), δ_l is the tolerable interference level to l PU, & γ_k are SUs' SNR requirements.

- By the SBP rank result, SDR is optimal when $K \leq 2$, $L = 1$ (≤ 2 SUs, 1 PU).

- CR BF design for the unicast case (see, e.g., **[Zhang-Liang-Cui'10]**):

$$\min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_K} \sum_{k=1}^{K} \|\boldsymbol{w}_k\|^2$$

$$\text{s.t. } \mathsf{SINR}_{\mathsf{SU},i} = \frac{\boldsymbol{w}_k^H \boldsymbol{R}_k \boldsymbol{w}_k}{\sum_{l \neq k} \boldsymbol{w}_l^H \boldsymbol{R}_k \boldsymbol{w}_l + \sigma_k^2} \geq \gamma_k, \; k = 1, \ldots, K,$$

$$\sum_{k=1}^{K} \boldsymbol{w}_k^H \boldsymbol{G}_l \boldsymbol{w}_k \leq \delta_l, \; l = 1, \ldots, L \quad \text{(IT constraints)}$$

- A separable QCQP with K variables and $K + L$ constraints.

- By the SBP rank result, SDR solves the problem if $L \leq 2$.

- Remark: For instant. CSIT with SUs, SDR can be shown to be optimal for any L. Or it can be reformulated, and then solved, by SOCP.

Part III: Frontier Development

Transmit Beamforming

- Transmit beamforming is now a key topic; see **[Gershman-Sidiropoulos-Shahbazpanahi-Bengtsson-Ottersten'10]**, **[Luo-Chang'10]** for review.

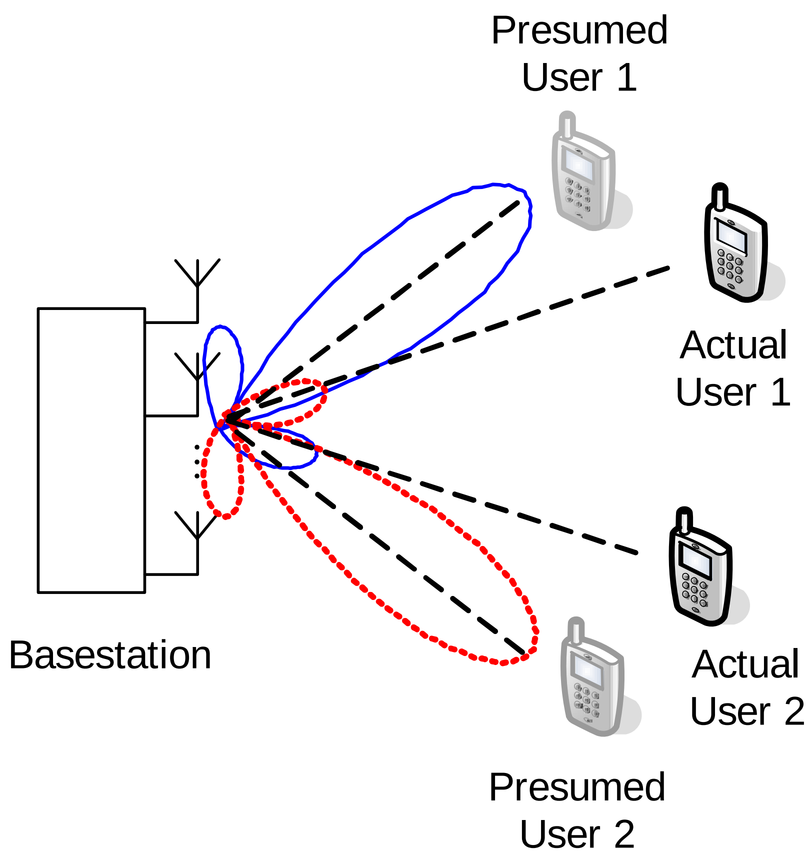
- Apart from standard transmit beamforming, we have seen numerous extensions:
  - one-way relay beamforming **[Fazeli-Dehkordy-Shahbazpanahi-Gazor'09]**, **[Chalise-Vandendorpe'09]**
  - two-way relay beamforming (a.k.a. analog network coding) **[Zhang-Liang-Chai-Cui'09]**
  - cognitive radio beamforming **[Zhang-Liang-Cui'10]**
  - multicell coordinated beamforming **[Bengtsson-Ottersten'01]**, **[Dahrouj-Yu'10]**
  - secrecy beamforming **[Liao-Chang-Ma-Chi'10]**, **[Li-Ma'11]**,

- Interestingly, all these beamforming problems turn out to be, or be closely related to, nonconvex QCQPs.

- And, as it turns out, SDR plays a key role.

Frontier Problem: Outage-Based Unicast Transmit Beamforming

CSIT is generally imperfectly known in practice.

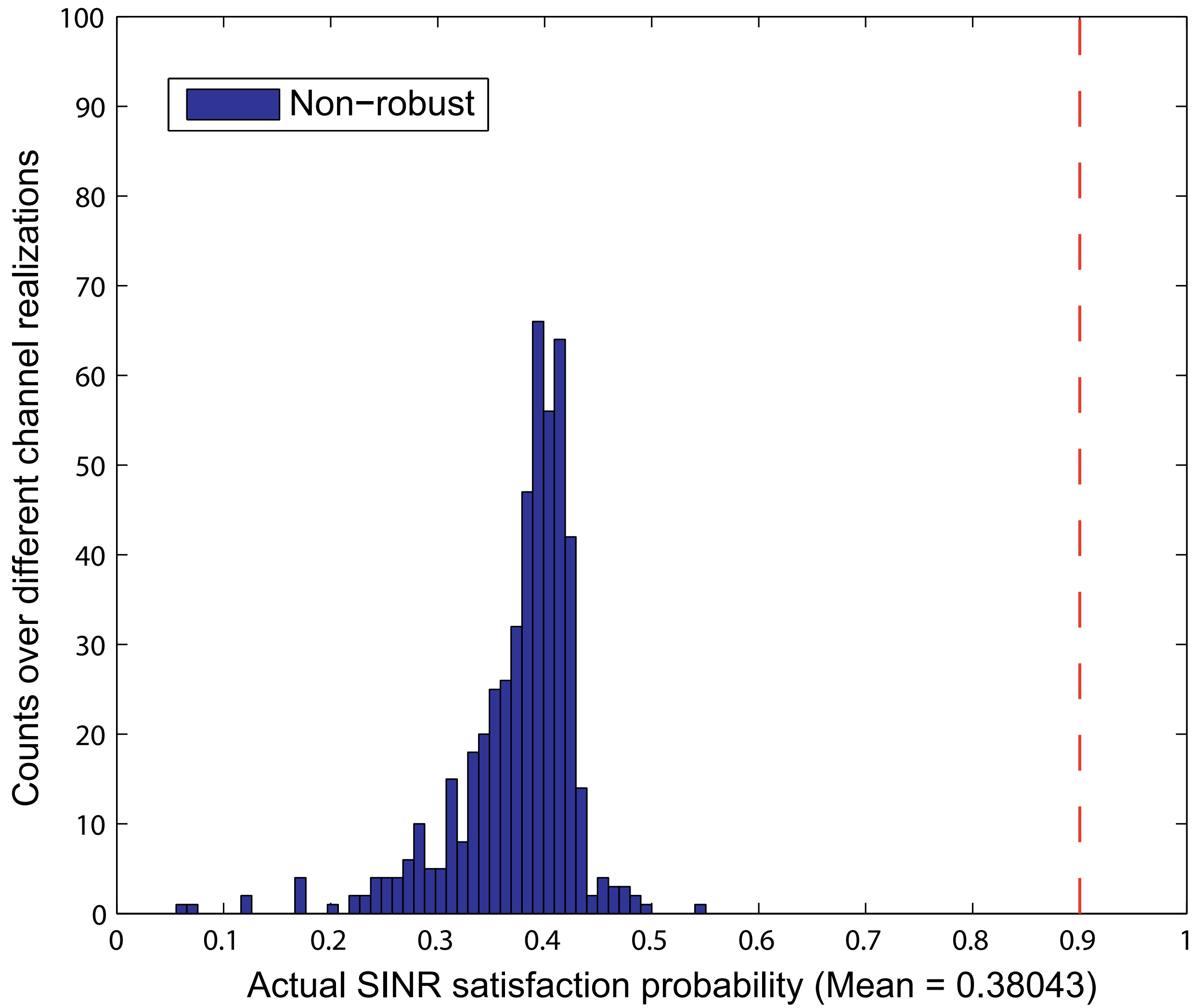


- Suppose that the presumed CSIT, $\{\boldsymbol{h}_i\}$, is inaccurate.

- If we directly substitute the presumed CSIT into the standard QoS-assured design

$$
\begin{aligned}
\min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_K \in \mathbb{C}^N} \quad & \textstyle\sum_{i=1}^K \|\boldsymbol{w}_i\|^2 \\
\text{s.t.} \quad & \frac{|\boldsymbol{w}_i^H \boldsymbol{h}_i|^2}{\sum_{l \neq i} |\boldsymbol{w}_l^H \boldsymbol{h}_i|^2 + \sigma_i^2} \geq \gamma_i, \\
& i = 1, \ldots, K,
\end{aligned}
$$

and run it, then the resultant design may have severe SINR outage.



Histogram of the actual SINR satisfaction probabilities of the non-robust QoS-assured design. $N_t = K = 3$; i.i.d. complex Gaussian CSI errors with zero mean and variance 0.002; $\gamma = 11$dB. The design has more than 50% outage most of the time.

Outage-Based Unicast Transmit Beamforming: Formulation

- Let us assume that $\boldsymbol{h}_i \sim \mathcal{CN}(\bar{\boldsymbol{h}}_i, \sigma_e^2 \boldsymbol{I})$, where $\bar{\boldsymbol{h}}_i$ is the presumed channel, and σ_e^2 is the CSI uncertainty variance.

- A meaningful, but very difficult, design problem:

$$\begin{aligned} \min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_K \in \mathbb{C}^N} \quad & \textstyle\sum_{i=1}^K \|\boldsymbol{w}_i\|^2 \\ \text{s.t.} \quad & \text{Prob}_{\boldsymbol{h}_i \sim \mathcal{CN}(\bar{\boldsymbol{h}}_i, \sigma_e^2 \boldsymbol{I})} \left\{ \frac{\boldsymbol{w}_i^H \boldsymbol{h}_i \boldsymbol{h}_i^H \boldsymbol{w}_i}{\sum_{l \neq i} \boldsymbol{w}_l^H \boldsymbol{h}_i \boldsymbol{h}_i^H \boldsymbol{w}_l + \sigma_i^2} \geq \gamma_i \right\} \geq 1 - \rho_i, \\ & \qquad i = 1, \ldots, K, \end{aligned}$$

where the ρ_i's are the maximum tolerable outage probabilities.

- The outage-based SINR constraints

$$\text{Prob}_{\boldsymbol{h}_i \sim \mathcal{CN}(\bar{\boldsymbol{h}}_i, \sigma_e^2 \boldsymbol{I})} \left\{ \frac{\boldsymbol{w}_i^H \boldsymbol{h}_i \boldsymbol{h}_i^H \boldsymbol{w}_i}{\sum_{l \neq i} \boldsymbol{w}_l^H \boldsymbol{h}_i \boldsymbol{h}_i^H \boldsymbol{w}_l + \sigma_i^2} \geq \gamma_i \right\} \geq 1 - \rho_i$$

can be rewritten as

$$\text{Prob}_{\boldsymbol{e}_i \sim \mathcal{CN}(\boldsymbol{0}, \sigma_e^2 \boldsymbol{I})} \left\{ (\bar{\boldsymbol{h}}_i + \boldsymbol{e}_i)^H \left(\frac{1}{\gamma_i} \boldsymbol{w}_i \boldsymbol{w}_i^H - \sum_{l \neq i} \boldsymbol{w}_l \boldsymbol{w}_l^H \right) (\bar{\boldsymbol{h}}_i + \boldsymbol{e}_i) \geq \sigma_i^2 \right\} \geq 1 - \rho_i.$$

- Challenges:
  - The probability on the LHS has no simple closed form expression.
  - The quadratic function

$$(\bar{\boldsymbol{h}}_i + \boldsymbol{e}_i)^H \left(\frac{1}{\gamma_i} \boldsymbol{w}_i \boldsymbol{w}_i^H - \sum_{l \neq i} \boldsymbol{w}_l \boldsymbol{w}_l^H \right) (\bar{\boldsymbol{h}}_i + \boldsymbol{e}_i)$$

    is indefinite (and hence nonconvex) in the design variables $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_K$.

Tackling the Nonconvexity: SDR

- Let us first do the thing we are good at — SDR.

- By SDR, we have

$$\mathrm{Prob}_{\boldsymbol{e}_i \sim \mathcal{CN}(\boldsymbol{0}, \sigma_e^2 \boldsymbol{I})} \left\{ (\bar{\boldsymbol{h}}_i + \boldsymbol{e}_i)^H \left(\frac{1}{\gamma_i} \boldsymbol{W}_i - \sum_{l \neq i} \boldsymbol{W}_l \right) (\bar{\boldsymbol{h}}_i + \boldsymbol{e}_i) \geq \sigma_i^2 \right\} \geq 1 - \rho_i.$$

- Now, the function

$$(\bar{\boldsymbol{h}}_i + \boldsymbol{e}_i)^H \left(\frac{1}{\gamma_i} \boldsymbol{W}_i - \sum_{l \neq i} \boldsymbol{W}_l \right) (\bar{\boldsymbol{h}}_i + \boldsymbol{e}_i)$$

  is linear in the variables $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_k$, which is good.

- However, the probability still does not admit a simple closed form expression.

Processing the Probabilistic Constraint: Convex Restriction

- Let

$$V_i(\{\boldsymbol{W}_j\}) = \text{Prob}_{\boldsymbol{e}_i \sim \mathcal{CN}(\boldsymbol{0}, \sigma_e^2 \boldsymbol{I})} \left\{ (\bar{\boldsymbol{h}}_i + \boldsymbol{e}_i)^H \left(\frac{1}{\gamma_i} \boldsymbol{W}_i - \sum_{l \neq i} \boldsymbol{W}_l \right) (\bar{\boldsymbol{h}}_i + \boldsymbol{e}_i) < \sigma_i^2 \right\}$$

be the violation probability. Recall that we want

$$V_i(\{\boldsymbol{W}_j\}) \leq \rho_i.$$

- It is not hard to see that V_i can be expressed as

$$V_i(\{\boldsymbol{W}_j\}) = \text{Prob}_{\boldsymbol{e} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I})} \left\{ \boldsymbol{e}^H \boldsymbol{Q} \boldsymbol{e} + 2\text{Re}\{\boldsymbol{e}^H \boldsymbol{r}\} + s < 0 \right\}$$

for some $\boldsymbol{Q}$, $\boldsymbol{r}$ and s that depend on $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_K$ and the index i. (Here and in the sequel, we drop the index i for notational simplicity.)

- To process the violation probability V_i, another idea is to find an efficiently computable convex function $f(\boldsymbol{Q}, \boldsymbol{r}, s, \boldsymbol{t})$, where $\boldsymbol{t}$ is an additional decision vector, such that

$$V_i(\{\boldsymbol{W}_j\}) = \text{Prob}_{\boldsymbol{e}\sim\mathcal{CN}(\boldsymbol{0},\boldsymbol{I})}\left\{\boldsymbol{e}^H\boldsymbol{Q}\boldsymbol{e} + 2\text{Re}\{\boldsymbol{e}^H\boldsymbol{r}\} + s < 0\right\} \leq f(\boldsymbol{Q}, \boldsymbol{r}, s, \boldsymbol{t}).$$

- Then, by construction, the convex constraint

$$f(\boldsymbol{Q}, \boldsymbol{r}, s, \boldsymbol{t}) \leq \rho \qquad \text{(CR-PC)}$$

serves as a sufficient condition for the probabilistic constraint

$$V_i(\{\boldsymbol{W}_j\}) \leq \rho \qquad \text{(PC)}$$

to hold. We call (CR-PC) a convex restriction of (PC).

Finding the Convex Restriction

- Can we find such a convex function? Does it even exist? The answer is: Yes! (And there are many such functions.)

- For instance, we can employ a Bernstein-type inequality **[Bechar2009]**, which states that

$$\text{Prob}_{\boldsymbol{e}\sim\mathcal{CN}(\mathbf{0},\boldsymbol{I})}\{\boldsymbol{e}^H\boldsymbol{Q}\boldsymbol{e} + 2\text{Re}\{\boldsymbol{e}^H\boldsymbol{r}\} + s < 0\} \leq e^{-T^{-1}(s)},$$

  where $T(\eta) = \text{Tr}(\boldsymbol{Q}) - \sqrt{2\eta}\sqrt{\|\boldsymbol{Q}\|_F^2 + \|\boldsymbol{r}\|^2} - \eta \max\{\lambda_{\max}(-\boldsymbol{Q}), 0\}$.

- Is the constraint

$$e^{-T^{-1}(s)} \leq \rho$$

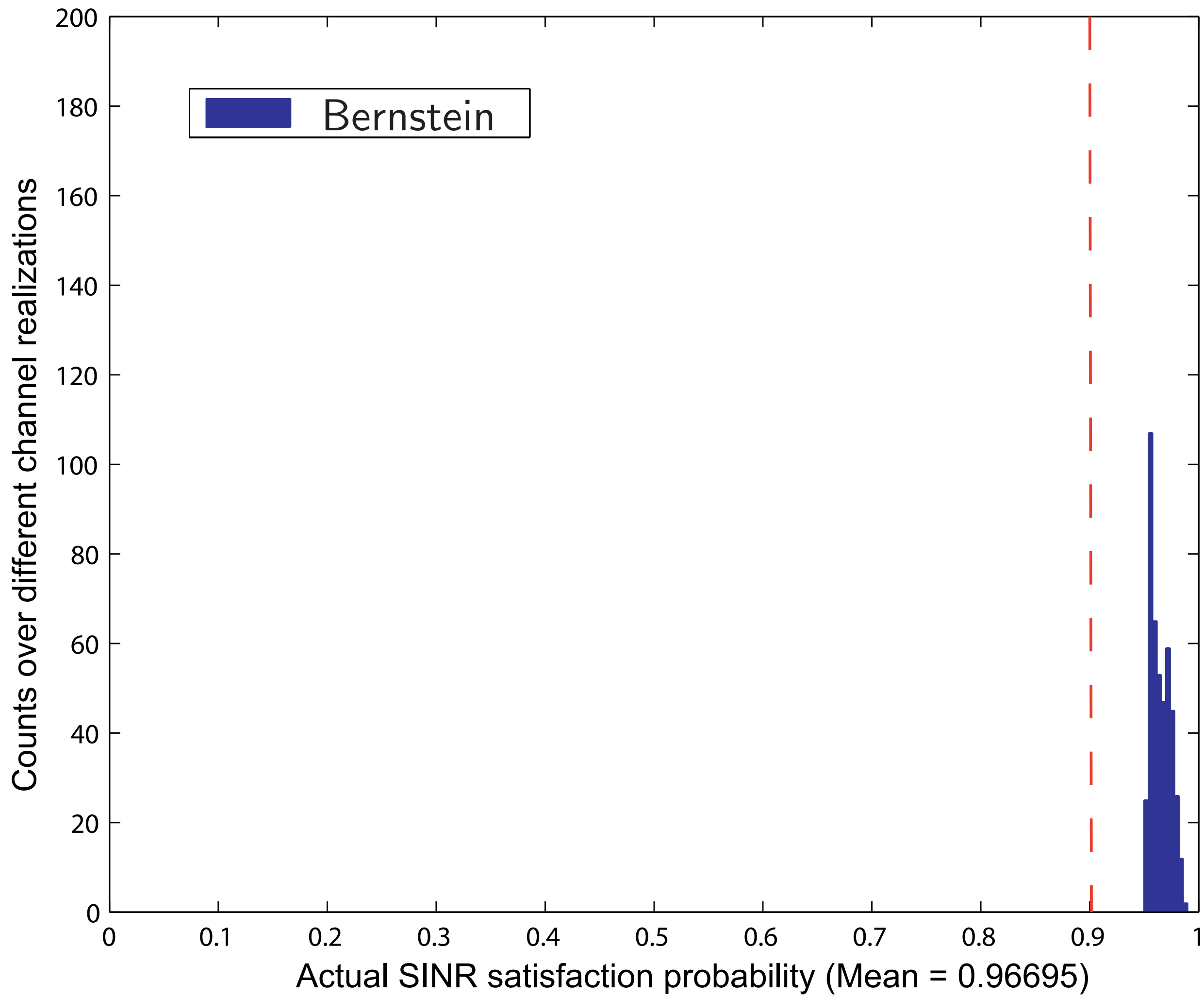
  convex? Yes! It is equivalent to

$$\begin{aligned}
&\text{Tr}(\boldsymbol{Q}) - \sqrt{-2\ln(\rho)} \cdot t_1 + \ln(\rho) \cdot t_2 + s \geq 0, \\
&\sqrt{\|\boldsymbol{Q}\|_F^2 + 2\|\boldsymbol{r}\|^2} \leq t_1, \\
&t_2\boldsymbol{I} + \boldsymbol{Q} \succeq \mathbf{0}, \\
&t_2 \geq 0.
\end{aligned}$$

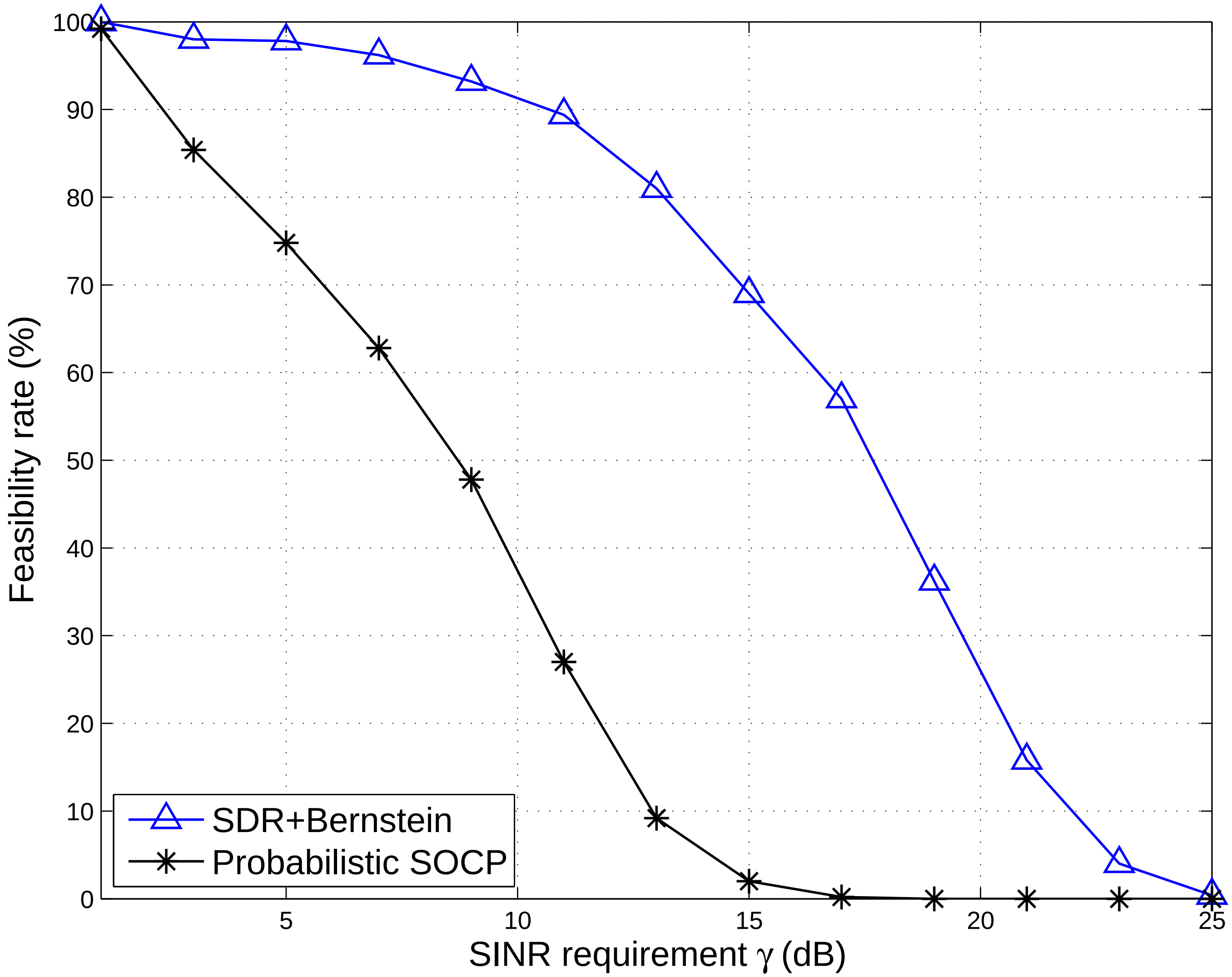
Putting Things Together: The Relaxation-Restriction Approach

- Applying the Bernstein-type inequality to the SDR'ed SINR constraints (with some additional work), a convex relaxation-restriction approximation is developed **[Wang-Chang-Ma-So-Chi'11]**.

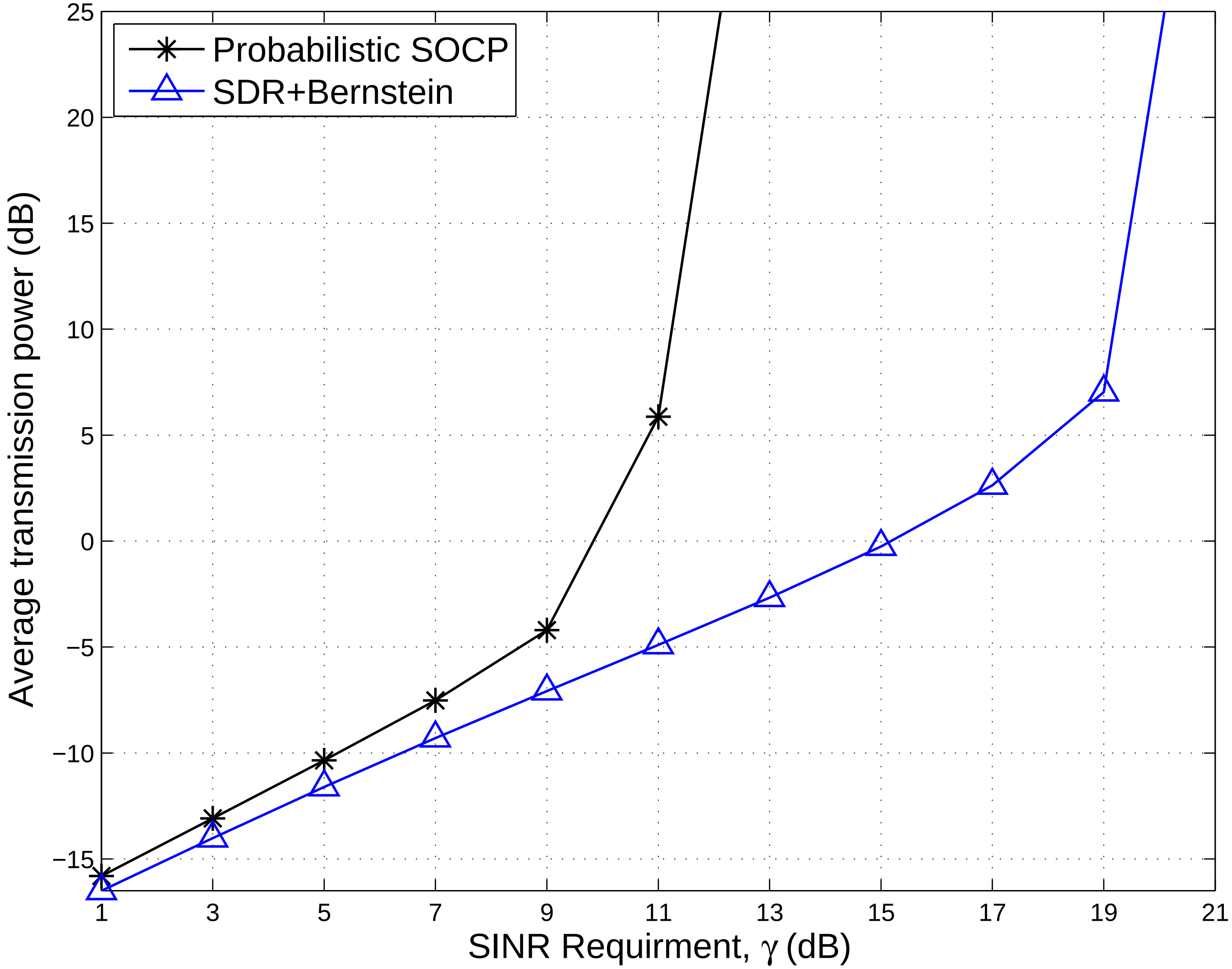
- **A mysterious finding in simulations:** rank-one SDR solution is obtained in almost all the problem instances!



Histogram of the actual SINR satisfaction probabilities of the proposed SDR+Bernstein method. $N_t = K = 3$; i.i.d. complex Gaussian CSI errors with zero mean and variance 0.002; $\gamma = 11$dB; $\rho = 0.1$ (90% SINR satisfaction).



Feasibility performance of the proposed method and the probabilistic SOCP method **[Shenouda-Davidson'08]**. $N_t = K = 3$; $\sigma_e^2 = 0.002$; $\gamma = 11$dB; $\rho = 0.1$ (90% SINR satisfaction).



Transmit power performance of the proposed method and the probabilistic SOCP method. $N_t = K = 3$; $\sigma_e^2 = 0.002$; $\rho = 0.1$ $(90\%$ SINR satisfaction).

References

[Lovász'79] L. Lovász, "On the Shannon capacity of a graph," *IEEE Trans. Inf. Theory*, 1979.

[Goemans-Williamson'95] M. X. Goemans and D. P. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problem using semi-definite programming," *J. ACM*, 1995.

[Tan-Rasmussen'01] P.H. Tan & L.K. Rasmussen, "The application of SDP for detection in CDMA," *IEEE JSAC*, 2001.

[Ma-Davidson-Wong-Luo-Ching'02] W.-K. Ma, T.N. Davidson, K.M. Wong, Z.-Q. Luo, & P.C. Ching, "Quasi-ML multiuser detection using SDR with application to sync. CDMA," *IEEE Trans. Signal Process.*, 2002.

[Bengtsson-Ottersten'01] M. Bengtsson & B. Ottersten, "Optimal & suboptimal transmit beamforming," in *Handbook of Antennas in Wireless Commun.*, L. C. Godara, Ed., CRC Press, 2001.

[Sidiropoulos-Davidson-Luo'06] N.D. Sidiropoulos, T.N. Davidson, & Z.-Q. Luo, "Transmit beamforming for physical layer multicasting," *IEEE Trans. Signal Proc.*, 2006.

[Biswas-Liang-Wang-Ye'06] P. Biswas, T.-C. Liang, T.-C. Wang and Y. Ye, "Semidefinite programming based algorithms for sensor network localization," *ACM Trans. Sensor Networks*, May 2006.

[Ma-Ching-Vo'04] W.-K. Ma, P.-C. Ching, and B.-N. Vo, "Crosstalk resilient interference cancellation in microphone arrays using capon beamforming," *IEEE Trans. Audio Speech Process.*, 2004.

[De Maio *et al.*'08] A. De Maio, S. De Nicola, Y. Huang, S. Zhang, and A. Farina, "Code design to optimize radar detection performance under accuracy and similarity constraints," *IEEE Trans. Signal Process.*, 2008.

[Chang-Luo-Wu *et al.*'08] T.-H. Chang, Z.-Q. Luo, X. Wu, C. Akgun, J. Vaughan, K. Ugurbil, and P.-F. V. de Moortele, "Transmit B1 shimming at high field with SAR constraints: A two stage optimization independent of the initial set of RF phases and amplitudes," in *Proc. 16th Intl. Society for Magnetic Resonance in Medicine*, May 2008.

[Quan-Ma-Cui-Sayed'10] Z. Quan, W.-K. Ma, S. Cui, and A. Sayed, "Optimal linear fusion for distributed spectrum sensing via semidefinite programming," *IEEE Trans. Signal Process.*, 2010.

[Li-Jiang'07] X. Li and H. Jiang, "Solving large-margin hidden Markov model estimation via semidefinite programming," *IEEE Trans. Audio Speech Lang. Process.*, Nov. 2007.

[Li-Bai-Ding'03] Q. Li, E.-W. Bai & Z. Ding, "Blind source separation of signals with known alphabets using ϵ-approximation algorithms," *IEEE Trans. Signal Process.*, 2003.

[Ma-Vo-Davidson-Ching'06] W.-K. Ma, B.-N. Vo, T.N. Davidson, & P.C. Ching, "Blind ML detection of orthogonal space-time block codes: Efficient high-performance implementations," *IEEE Trans. Signal Process.*, 2006.

[Ma-Ching-Ding'04] W.-K. Ma, P.C. Ching, & Z. Ding, "SDR based multiuser detection for MPSK multiuser systems," *IEEE Trans. Signal Process.*, 2004.

[Ma-Su-Jaldén-Chang-Chi'09] W.-K. Ma, C.-C. Su, J. Jaldén, T.-H. Chang, & C.-Y. Chi, "The equivalence of semidefinite relaxation MIMO Detectors for higher-order QAM," *IEEE J. Sel. Topics Signal Process.*, Dec 2009.

[Steingrimsson-Luo-Wong'03] B. Steingrimsson, Z.-Q. Luo, and K. M. Wong, "Soft quasi-maximumlikelihood detection for multiple-antenna wireless channels," *IEEE Trans. Signal Process.*, Nov. 2003.

[Kisialiou-Luo-Luo'09] M. Kisialiou, X. Luo, and Z.-Q. Luo, "Efficient implementation of quasi-maximum-likelihood detection based on semidefinite relaxation," *IEEE Trans. Signal Process.*, 2009.

[Wai-Ma-So'11] H.-T. Wai, W.-K. Ma, and A. M.-C. So, "Cheap semidefinite relaxation MIMO detection using row-by-row block coordinate descent," in *ICASSP 2011*, 2011.

[Jaldén-Ottersten'08] J. Jaldén and B. Ottersten, "The diversity order of the SDR detector," *IEEE Trans. Inf. Theory*, 2008.

[Kisialiou-Luo'10] M. Kisialiou and Z.-Q. Luo, "Probabilistic analysis of semidefinite relaxation for binary quadratic minimization," *SIAM Journal on Optimization*, 2010.

[So'10] A. M.-C. So, "Non-asymptotic performance analysis of the semidefinite relaxation detector in digital communications," Preprint, 2010.

[Pataki'98] G. Pataki, "On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues," *Math. Oper. Res.*, 1998.

[Ye-Zhang'03] Y. Ye and S. Zhang, "New results on quadratic minimization," *SIAM J. Optim.*, 2003.

[Huang-Zhang'07] Y. W. Huang and S. Zhang, "Complex matrix decomposition and quadratic programming," *Math. Oper. Res.*, Aug. 2007.

[Huang-Palomar'09] Y. Huang and D. Palomar, "Rank-constrained separable semidefinite program with applications to optimal beamforming," *IEEE Trans. SP*, 2010.

[Gershman-Sidiropoulos-Shahbazpanahi-Bengtsson-Ottersten'10] A. B. Gershman, N. D. Sidiropoulos, S. Shahbazpanahi, M. Bengtsson, and B. Ottersten, "Convex optimization-based beamforming: From receive to transmit and network designs," *IEEE Signal Processing Magazine*, Special Issue on Convex Optimization for Signal Processing, 2010.

[Luo-Chang'10] Z.-Q. Luo and T.-H. Chang, "SDP relaxation of homogeneous quadratic optimization: Approximation bounds and applications," in *Convex Optimization in Signal Processing and Communications*, 2010.

[Fazeli-Dehkordy-Shahbazpanahi-Gazor'09] S. Fazeli-Dehkordy, S. Shahbazpanahi, and S. Gazor, "Multiple peer-to-peer communications using a network of relays," *IEEE Trans. Signal Process.*, Aug. 2009.

[Chalise-Vandendorpe'09] B.K. Chalise and L. Vandendorpe, "MIMO Relay Design for multipoint-to-multipoint communications with imperfect channel state information," *IEEE Trans. Signal Process.*, July 2009.

[Zhang-Liang-Chai-Cui'09] R. Zhang, Y.-C. Liang, C.-C. Chai, and S. Cui, "Optimal beamforming for two-way multi-antenna relay channel with analogue network coding," *IEEE J. Select. Areas Commun.*, June 2009.

[Zhang-Liang-Cui'10] R. Zhang, Y. C. Liang, and S. Cui, "Dynamic resource allocation in cognitive radio networks," *IEEE Signal Processing Magazine*, Special Issue on Convex Optimization for Signal Processing, May 2010.

[Dahrouj-Yu'10] H. Dahrouj and W. Yu, "Coordinated beamforming for multicell multi-antenna wireless system," *IEEE Trans. Wireless Comm.*, May 2010.

[Liao-Chang-Ma-Chi'11] W.-C. Liao, T.-H. Chang, W.-K. Ma, and C.-Y. Chi, "QoS-based transmit beamforming in the presence of eavesdroppers: An artificial-noise-aided approach," *IEEE Tans. Signal Process.*, Mar 2011.

[Li-Ma'11] Q. Li and W.-K. Ma, "Optimal and robust transmit designs for MISO channel secrecy by semidefinite programming," *IEEE Trans. Signal Process.*, Aug. 2011.

[Phan-Vorobyov-Sidiropoulos-Tellambura'09] K. Phan, S. Vorobyov, N. Sidiropoulos, and C. Tellambura, "Spectrum sharing in wireless networks via QoS-aware secondary multicast beamforming," *IEEE Trans. Signal Process.*, June 2009.

[Wyner'75] A. D. Wyner, "The wiretap channel," *The Bell System Technical Journal*, vol. 54, October 1975, pp. 1355–1387.

[Nego-Goel'05] R. Negi and S. Goel, "Secret communication using artificial noise," in *IEEE Vehicular Technology Conference*, Sept. 2005, pp. 1906-1910.

[Li-Ma'11-B] Q. Li and W.-K. Ma, "A robust artificial noise aided transmit design for MISO secrecy," in *ICASSP 2011*.

[Li-Ma-So'11] Q. Li, W.-K. Ma, and A. M.-C. So, "Safe convex approximation to outage-based MISO secrecy rate Optimization under imperfect CSI and with artificial noise," to appear in *Asilomar 2011*.

[Calafiore-Campi'05] G. Calafiore and M. C. Campi, "Uncertain convex programs: Randomized solutions and confidence levels", *Math. Prog.*, 2005.

[Bechar'09] I. Bechar, "A Bernstein-type inequality for stochastic processes of quadratic forms of Gaussian variables," *Preprint*, 2009.

[Wang-Chang-Ma-So-Chi'11] K.-Y. Wang, T.-H. Chang, W.-K. Ma, A. M.-C. So, and C.-Y. Chi, "Probabilistic SINR constrained robust transmit beamforming: A Bernstein-type inequality based conservative approach," in *ICASSP 2011*, 2011.

[Shenouda-Davidson'08] M. B. Shenouda and T. N. Davidson, "Probabilistically-constrained approaches to the design of the multiple antenna downlink," in *Proc. 42nd Asilomar Conference 2008*.